SCIENCE AND ART

CHEMISTRY

PRACTICAL PLANE AND SOLID

GEOMETRY.

Collins' Elementary Science Series.

PRACTICAL PLANE AND SOLID GEOMETRY:

PREPARED SPECIALLY FOR

STUDENTS IN SCIENCE CLASSES IN CONNECTION WITH THE SCIENCE AND ART DEPARTMENT.

BY

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PREFACE.

GEOMETRY is one of the most attractive studies to artizans, on account of its practical utility. But attention has been heretofore confined principally to one branch of the subject—Plane Geometry. It is only of late years that Solid Geometry has been brought so extensively into notice in England.

The Science and Art Department has from year to year more strongly advocated the study of the latter branch of the subject, by rejecting greater numbers of those candidates, at their examinations of science classes, who have failed to come up to the gradually increasing standard of excellence in knowledge of its principles.

Foreign nations have been before us in this matter; and the recent Educational Exhibitions have shown that a knowledge of Solid Geometry is considered indispensable to the well-educated foreigner.

There are few English works upon this subject; and those which do exist are either too learned for the ordinary reader, or they are exceedingly expensive. This little treatise has been prepared expressly for those who are studying Geometry in classes in connection with the Science and Art Department. The aim of the writer, who has taught the subject to large classes of artizans for several years, has been to show, as far as possible, the principles upon which constructions are based, thereby helping the student to avoid the system of "cram," of which examiners so justly complain.

He should not rest satisfied until he can understand the "why" and "wherefore" of the point he is studying, and can reason out for himself any necessary deductions therefrom.

It is understood that, in future, candidates' attention must be confined almost entirely to Solid Geometry.

This book contains rather more than is necessary to correctly work the elementary papers set at the South Kensington Examinations.

H. A.

ISLINGTON SCIENCE SCHOOL,
December, 1872.

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GEOMETRY.

CHAPTER I.

INTRODUCTION.

GEOMETRY is divided into two distinct branches—**Theoretical** and **Practical.** The former proves the principles of the science, whilst the latter applies those principles to construction. It is our duty to consider only the practical branch of the subject, although reference will be made to Euclid, and proofs of constructions given where advisable; so that the student may, if he choose, more clearly understand the solution of a problem by investigating the principles upon which it is based.

Practical Geometry is subdivided into two branches—**Plane** and **Solid.** The former describes the construction and properties of lines and figures, whilst the latter treats of the delineation of solid bodies upon plane surfaces.

It is necessary that the student should be provided with the following materials, to enable him to work out for himself the problems contained in this book:—

(1.) A DRAWING-BOARD. This should be quite square at its corners, and present a perfectly level surface. The size would, of course, depend upon the kind of work to be done; but a board 22 inches by 17 inches will be found very generally useful.

(2.) A T SQUARE. By means of this instrument, perpendicular and horizontal lines can be drawn parallel to the edges of the board; and if the head be so

constructed as to turn upon the blade, lines at any angle with these perpendiculars, &c., can be obtained. The edge of the blade should be bevelled, as the instrument will not then throw a shadow where the line is to be drawn.

(3.) 2 SET-SQUARES (60° and 45°). These consist of two triangular pieces of wood or vulcanite. Those having angles of 60° and 45° are the most convenient. By means of these, lines can be drawn perpendicular to each other, or parallels in any direction can be determined. Thus, in figure 1, one set-square, *b*, rests against the edge of another, *a*. It will be seen that by sliding the former along the latter to other positions, as *c* and *d*, parallels or perpendiculars to the edge of the set-square *a*, can be drawn at any distance from each other.

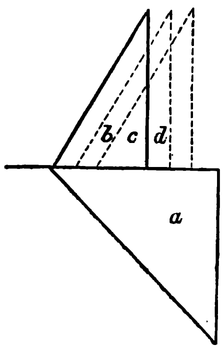


Fig. 1.

(4.) A SET OF MATHEMATICAL INTRUMENTS, which should comprise, at least—a compass, with moveable pen and pencil legs; a pair of dividers; bow pen and pencil compasses, to describe small circles and arcs; and a ruling pen. Indian ink should be used with the instruments, because it will not corrode them. After using, they should be wiped quite clean, to preserve them from rust.

(5.) A PROTRACTOR. This is an instrument used for setting out angles. It is made in several forms; but the most convenient for the student is the six-inch flat rule, with 180° marked round three of its edges. The method of using it is as follows :—Suppose an angle of 40° is to be made with a given line, A B, at some point, A, in it. The unmarked edge of the instrument should be so placed as to coincide with the line A B, the centre of that edge resting upon A. Then, if the

required angle is to open from right to left, the numbers of the degrees upon the protractor must be read in that direction; and at the required 40°, a mark should be

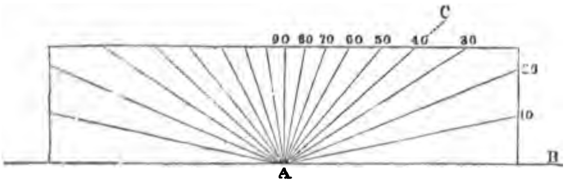


Fig. 2.

made upon the paper. Then, by removing the instrument, and joining the point found to A, an angle of 40° with the line A B will be determined.

(6.) PAPER AND PENCILS. *Cartridge Paper* is the cheapest and best which can be used for geometrical drawing purposes, as it is stout enough to prevent the points of the instruments from penetrating, if they are used carefully. The pencils should be those marked "H." and "H. B.;" the former, for what are termed construction lines, and the latter for completed figures, which should be drawn in firm dark lines.

(7.) DRAWING PINS. These are required to keep the paper in a fixed position upon the board. The best are those which have the pins soldered into the heads, but not penetrating quite through them. By using this kind, the annoyance of the pin coming through and pricking the finger, or unscrewing when taken out, is avoided.

CHAPTER II.

DEFINITIONS AND ELEMENTARY PROBLEMS.

A **point** has neither length, breadth, nor thickness. It merely denotes a position, and is shown in geometrical drawings thus, —⊙A.

A **straight line** has length, but not breadth nor thickness. It is the nearest distance between two given points.

An **angle** is the inclination to each other of two straight lines which meet in a point.

When one straight line meeting another straight line makes the angles on either side of it equal to one another, each of these angles is a **right angle**; and the lines are said to be mutually **perpendicular**. (Euclid, Bk. I., Def. 11.)

An angle is **acute**, when smaller than a right angle, and **obtuse** when greater.

The **complement** of an angle is that which it requires to complete a right angle.

The **supplement** of an angle is that which it requires to complete two right angles.

A **triangle** is a figure enclosed by three straight lines. When these lines are equal, the triangle is **equilateral**; when two of them only are equal, it is **isosceles**.

A **right-angled** triangle has one of its angles a right angle.

A **quadrilateral** figure is enclosed by four straight lines.

A **square** is a quadrilateral figure having all its sides equal and all its angles right angles.

A **rectangle** has two pairs of equal sides, and all its angles right angles.

A **parallelogram** is a figure having two pairs of parallel sides.

A **rhombus** is a quadrilateral figure having all its sides equal, but two of its angles acute.

The **diagonal** of a rectilineal (straight-lined) figure is the line which joins two opposite angular points.

A **polygon** is a figure having many sides. Polygons are **regular** or **irregular**, according as their sides are equal or **unequal**. Special names are given to polygons according to the number of their sides. Thus—

Pentagon, .	5 sides.	Nonagon, .	9 sides.
Hexagon, .	6 "	Decagon, .	10 "
Heptagon, .	7 "	Undecagon, .	11 "
Octagon, . .	8 "	Duodecagon, .	12 "

A **circle** is a space enclosed by a line which at all parts is equidistant from a fixed point, called the **centre**. The boundary line, or circumference, is also called a circle.

The straight line passing through the centre, and meeting the circumference in two points, is the **diameter**.

A **radius** is half a diameter.

Any part of the circumference of a circle is called an **arc**.

The straight line joining the extremities of an arc is called the **chord** of that arc.

A **semicircle** is half a circle.

The space enclosed by an arc and its chord is a **segment**.

The space enclosed by two radii and the intercepted arc is a **sector**.

A straight line touching a circle, but not cutting it, is a **tangent** to that circle.

A tangent is perpendicular to the radius which passes through the point of contact.

The mark (") means inches—thus 3·7" means 3·7 inches.

A line is named by letters placed at its extremities, as line A B

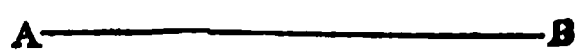


Fig. 3.

An angle is named either by a single letter placed at the intersection of the two lines forming it, or by three letters, the middle one being that described above. Thus, angle B, or A B C.

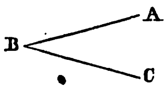


Fig. 4.

A figure is named by letters placed at its angular points, as figure A B C.

Fig. 5.

A circle is named by a letter placed at its centre.

Parts of an inch are given as decimal fractions. Thus, 6·5" means six inches and five tenths of another inch; 3·25" means three inches and twenty-five hundredths of another inch. One-half is represented by ·5, one quarter by ·25 and three quarters by ·75.

Where the figures in this work do not agree with the dimensions given in the problems, the scale is ½.

The problems of this chapter are to be worked with compass and ruler only. Perpendiculars and parallels are to be constructed by rule, and not drawn mechanically by aid of the T square or set-squares. Angles, too, are to be determined geometrically—that is, without the aid of the protractor.

Problem I.

To divide a finite straight line into two equal parts.

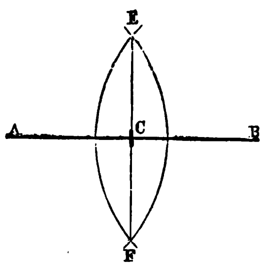


Fig. 6.

Let A B be the given straight line. With A and B, in turn, as centres, and with a radius obviously larger than half the line, describe arcs intersecting in E and F. Join E F. Then the point C, where E F meets A B, is the centre of the line. By extending this process and bisecting each half again, the line can be divided into four equal parts.

Problem II.

At the given points A *and* B, *in the straight line* C D, *to erect perpendiculars.*

On either side of the point A mark off equal distances, as A E, A F. With E and F as centres—radius, E F, describe arcs intersecting in G. Join G A.

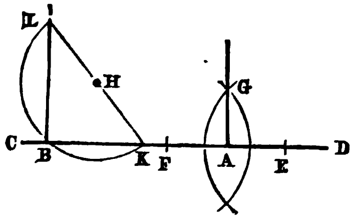


Fig. 7.

The above construction would not be convenient for determining a line perpendicular to C D, and to pass through B, as that point is so near the extremity of the line. In this case, take *any* point, H, as centre, and describe an arc, K B L, passing through the point B. Join K H, and produce it beyond H, until it meets the arc in L. Join L B.

Problem III.

Through the given points A *and* B, *to draw lines perpendicular to* C D.

In this case the given points are *without* the given line. With A as centre, draw an arc, E L F, which will cut C D in two points E and F. Bisect E F (Prob. I.) in the point G. Then A G is the required perpendicular.

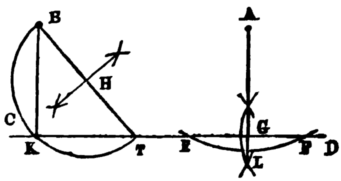


Fig. 8.

Another method is required when the given

point, as B, is nearly over the extremity of the line. Draw any line, B T, intersecting C D in F. Bisect B T (Prob. I.), and, with H as centre—radius H B—describe the arc B K T, intersecting C D in K. Join B K.

PROBLEM IV.

To bisect a given angle, B A C.

On A B and A C mark off equal distances, A E and A F.

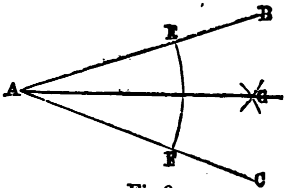


Fig 9.

With E and F as centres, and with a radius equal to more than half the distance E F, describe arcs intersecting in G. Then, a line, A G, will bisect the angle.

PROBLEM V.

To draw a line, making an angle of 60°, with the given line A B, at the point A.

Before commencing this problem, the student will require a little instruction as to the conventional method of measuring angles. If he will take his compass, and keeping one leg stationary, will revolve the other about the hinge as a centre, he will notice that the opening between the legs will increase, until the two form one straight line. If this revolution could be continued far enough, a complete circle would be generated.

In England it is agreed, that the whole revolution shall be supposed to be divided into 360 equal steps, each step being a degree, written thus °. Consequently, when the moveable leg has made one quarter of a revolution, it will have travelled through 90°. When half a revolu-

tion has been made, a straight line is formed, which theoretically is an angle of 180°; but in practical geometry no angle, greater than 179° is referred to. Any angle, therefore, is determined by the number of degrees which it contains. In the problem before us, we have to make an angle of 60°, which it is readily seen is one-sixth part of 360°.

With A as centre, draw any arc, C D, and as *the radius of a circle stepped round the circumference will divide it into six equal parts*, mark off C D, equal to the radius employed. Join A D, and D A B is the required angle. The student has now learned how to construct angles of 90° and 60°. (Prob. II. and III.) He can also bisect an angle. By the proper use of the constructions already described, many other angles can be determined. Thus, an angle of 30° is obtained by bisecting an angle of 60°; 45°, by bisecting 90°; 135°, by adding 90° to 45°; 120°, by doubling 60°; 15°, by bisecting 30°; and 75°, by adding 15° to 60°.

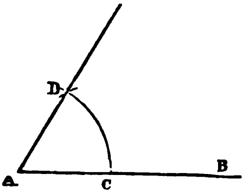


Fig. 10.

Problem VI.

To draw a line parallel to the given line, A B, *at a distance of* 1·2″ *from it.*

At any two points C and D, in the given line A B, construct two perpendiculars, 1·2″ in length (Prob. II). Join their extremities, E and F, and the required line will be determined.

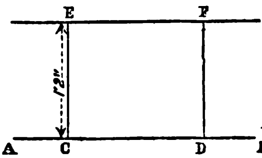


Fig. 11.

1 E. B

Problem VII.

Through a given point, C, to draw a line parallel to a given line, A B.

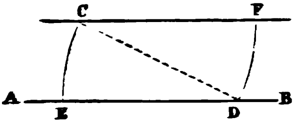


Fig. 12.

Take any point D, in A B, as centre, and with radius C D draw the arc C E. With C as centre, and radius C D, draw the arc D F. Make D F equal to C E, and join C F. Then C F is the required parallel.

Note.—The angles F C D, E D C, are equal, and are called alternate angles.

Problem VIII.

Through a given point, C, *to draw a line meeting a given line* A B, *at an angle of* 60°.

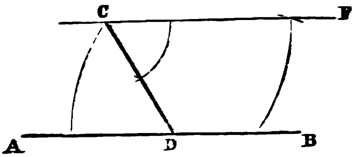


Fig. 13.

Through C, draw a line, C F, parallel to A B (Prob. VII.), and at the point C make C D so that the angle F C D shall be equal to the given one (60°). Then C D A will be an angle of 60°. (See note on Prob. VII.)

Problem IX.

On a given straight line, A B, *to construct an Equilateral Triangle, a Square, and a Hexagon.*

With A and B as centres, radius A B, describe arcs

intersecting in C. Join A C, B C. A B C is an equilateral triangle.

At B erect B E perpendicular to A B (Prob II.), making it equal in length to A B. With A and E as centres, radius A B, describe arcs intersecting in D. Join D E, A D, and A B E D is the required square.

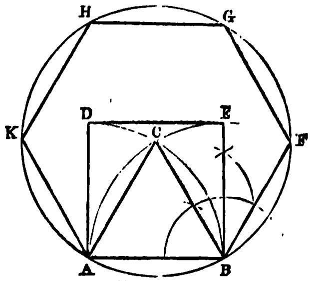


Fig. 14.

With C as centre, radius A C, describe the circle A B G. Then A B stepped round the circle will divide it into six equal parts in A B F G H and K. Join these points, as in the diagram, and the required hexagon will be determined.

PROBLEM X.

To inscribe in a given circle a Square and an Octagon.

One figure is said to be inscribed in another when all the angular points of the former are in the boundary line of the latter.

Draw two diameters, A C and B D, perpendicular to each other. Join A B, B C, C D, and D A. Then A B C D is the required square. Bisect the arcs C D and A D (Prob. I.) in the points H and K. Draw lines from H and K to pass through the centre, E, meeting the circle in F and G. Join the points thus found, and the required octagon will be determined.

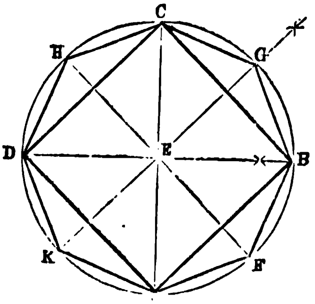


Fig. 15.

Problem XI.

In a given circle, to inscribe a Hexagon and an Equilateral Triangle.

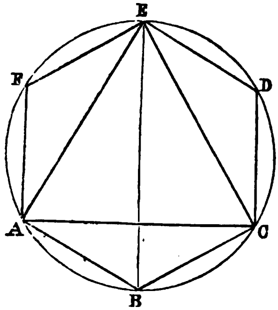


Fig. 16.

As the radius divides a circle into six equal parts, the hexagon is completed by joining the points of division. If the alternate points only be joined, an inscribed equilateral triangle will be determined.

Problem XII.

To construct a Rectangle, the diagonal of which shall be 2" long, one side being ·8" long.

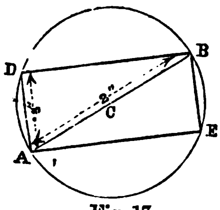


Fig. 17.

Draw a line, A B, 2″ long, and bisect it (Prob I) in the point C. With C as centre, and A C as radius, describe a circle. Mark off A D and B E, each ·8" long, and join A D, B D, A E, and B E. Then A B D E is the required rectangle.

Problem XIII.

To construct a Rhombus, having one of its angles 45°, its sides being 1·5″ long.

Draw A B 1·5″ long, and at A make an angle of 45°,

by first constructing a right-angle, and then bisecting it. Make A D equal to A B, and with B and D as centres, radius A B, describe arcs intersecting in C. Then A B C D is the required rhombus.

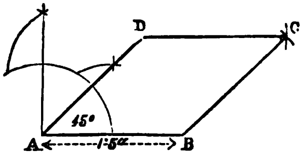


Fig. 18.

PROBLEM XIV.

In a given straight line, A B, *to find a point,* F, *equidistant from two given points,* C *and* D.

Join the given points. Bisect C D in E (Prob. I.),

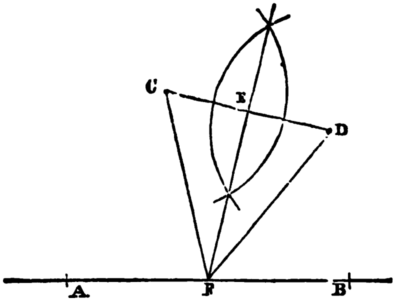


Fig. 19.

and draw E F perpendicular, meeting the line A B in F. Then the distances, F C, F D, will be equal.

PROBLEM XV.

To describe a circle which shall pass through three given points, A, B *and* C.

Join A B. Bisect it by the perpendicular, D E. Join

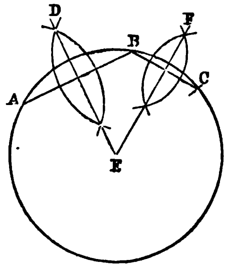


Fig 20.

B C, and bisect it by the perpendicular F E, intersecting D E in the point E. Then, with E as centre, draw a circle passing through either of the given points, and it will be found to contain the other two.

Note.—The centre of a circle can be determined by assuming any three points in its circumference, and proceeding as above.

Problem XVI.

To draw two Tangents to the given circle, C, each passing through one of the given points, A and B.

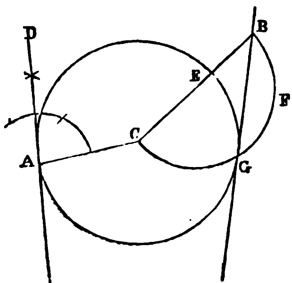


Fig 21.

(1.) Join A C, and at the point A draw A D perpendicular to it. This is the required tangent, as lines which touch circles are perpendicular to the radii at the points of contact.

(2.) Join B C, and bisect it in E. With E as centre, describe the semi-circle, B F C, meeting the circle in G. Then B G is the required tangent.

EXERCISES.

1. At a point A, in an indefinite line A B, construct the following angles, without using a protractor or scale of chords:—B A C = 15°, B A D = 37½°, B A E = 75°, B A F = 135°.

2. Draw a straight line 6" long, and divide into 32 equal parts, by continual bisection.

3. Make any triangle, and draw a line perpendicular to the base, and passing through the apex.

4. Draw a square, and by means of parallels to its sides—1" away—construct another one.

5. Draw an equilateral triangle, and on its three sides construct respectively, a square, a hexagon, and a rhombus with an angle of 60°.

6. Draw a line 3·5" long, and at one extremity erect a perpendicular 1·75" long. From the top of the perpendicular draw a line to make an angle of 30° with the given line.

7. Draw a circle of 1·75" radius; divide it into 6 equal parts. At each of the points of division draw a line tangent to the circle.

8. Draw a circle, and determine (as if unknown) its centre.

CHAPTER III.

PROBLEMS TESTING NEATNESS AND ACCURACY OF DRAWING.

THE problems of this chapter will not require a very extensive knowledge of geometry. They are intended to train the student to habits of neatness and exactness, without which his constructions can be of little value. A few hints are given, which have been found very useful by the writer.

If a line is intended to pass through a point, be careful neither to draw it a little above or below, nor to the right or to the left of that point.

Draw *from* a point, not *to* it.

Do not let the intersection of your arcs be too acute, as the exact point where the lines cut each other, in such a case, is not easily discerned.

Measure long distances in preference to short ones, where practicable. Thus, if an unequally divided line is to be copied, measure off upon an equal line the length of the greater segment in preference to that of the lesser.

In taking degrees from a protractor, be very careful to set the instrument exactly, and make the pencil mark in the same direction as that shown upon the protractor.

In drawing parallels with the T square and set squares, be sure that the fixed instrument is in its correct place, and that the moveable one has its edge close to that of the former.

So place your straight-edge that the part you rule by may not be in shadow.

Make your constructions as large as possible; and where dimensions are given, do not alter them.

A chisel-shaped point for the pencil is best, as it can be kept well up to the edges of your rulers.

Perpendiculars and parallels may, in all future problems, be mechanically determined by the aid of your T square, &c.

Problem XVII.

Make any four-sided figure, A B C D, and mark any point E, within it. Join E A, E B, E C, E D. Divide E A into 3 equal parts, by trial with dividers, in the points 1, 2. By the aid of your set squares, draw lines through 1 and 2 parallel to A B. If the construction be accurate, the line B E will be also divided equally

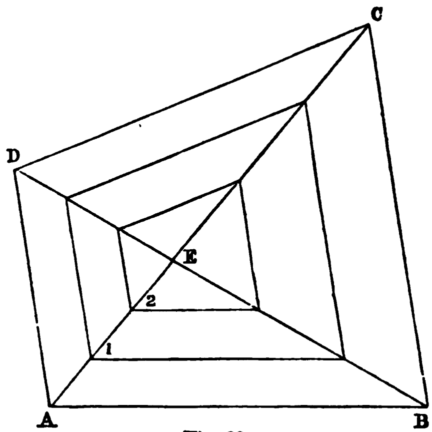


Fig. 22.

into 3 parts. Again, through the points of division of B E, draw parallels to B C, and so continue round the figure until they meet the first divided line, A E, in points 1 and 2. The exercise is rendered still more useful as a test of accurate drawing, by taking a greater number of sides for the first figure.

PROBLEM XVIII.

Draw three equal Circles of ·75" radius, each touching the other two.

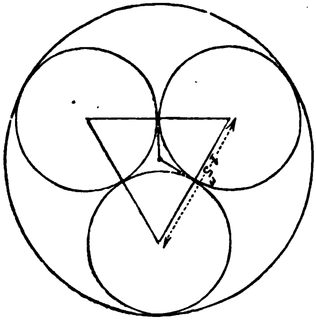


Fig. 23.

The centres of these circles will be the angular points of an equilateral triangle of 1·5" side. Construct this figure, and draw the circles. To test the accuracy of the drawing, bisect two sides of the equilateral triangle, and join their middle points to the opposite corners. The point where these lines intersect can be used as the centre of a circle circumscribing those first drawn.

PROBLEM XIX.

Draw any straight line, A B, and mark upon it any number of equal distances about ·5" long, as 1, 2, 3, 4, 5, &c. Place the T square so that its edge may coincide

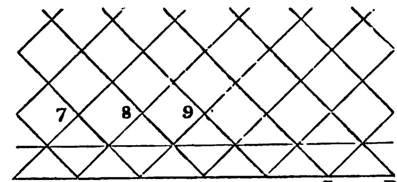


Fig. 24.

with A B, and by aid of the 60° set-square, draw parallel lines through each of the points of division. Reverse

the set square, and draw parallel lines in the opposite direction through the points of division. It is easy then to test the accuracy of the construction by comparing parts which should be equal in length on this trellis-work pattern. If correct, a straight line will pass through the points 7, 8, 9, &c.

PROBLEM XX.

Construct a six-sided Polygon, A B C ... F, *from the following conditions:—*

Sides.	*Angles.*
$A\ B = 1{\cdot}5''$	$A\ B\ C = 100°$
$B\ C = 2''$	$B\ C\ D = 110°$
$C\ D = 2{\cdot}25''$	$C\ D\ E = 120°$
$D\ E = 2{\cdot}5''$	$D\ E\ F = 130°$.
$E\ F = 3''$	

Write down the length of the side A F, *and the magnitude of the angles* E F A, F A B, (*Science Exam.*)

The line A B (1·5″ long) must be drawn first; the angle A B C (100°) must then be laid off from the pro-

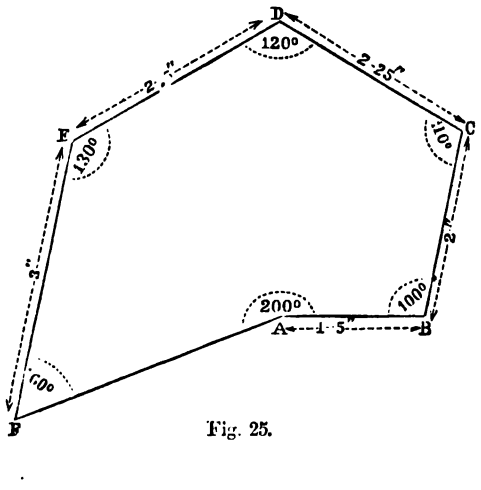


Fig. 25.

tractor. Make B C (2″) and the angle, B C D (110°) determines the direction of C D. By proceeding in this way, all the sides of the figure can be constructed, except A F, which should be 3″ long. The angles, E F A and F A B, will be found by the protractor to contain 60° and 200° (a re-entrant angle) respectively.

The sums of the angles should satisfy the following equation, n being the number of sides—

$$S = (2n - 4)90^{\circ}. \quad (\textit{Euclid}, \text{ Bk. I., Def. 32.})$$

Problem XXI.

Draw a square of 2″ side. On each diagonal as a base, draw two equilateral triangles. In each of these four triangles inscribe a circle.

Make the square A B C D (Prob. IX.), and the diagonals A C and B D. On both sides of each of these construct

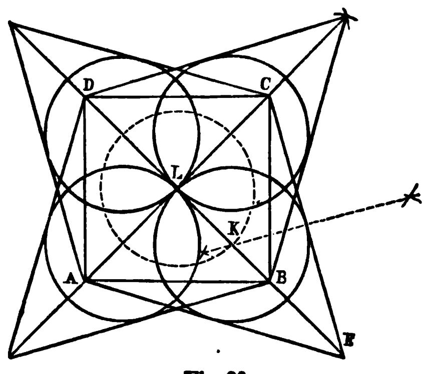


Fig. 26.

an equilateral triangle, (Prob. IX.) Produce the diagonals until they meet the vertices (highest points) of the triangles, and bisect one of the sides as E C. Then

K is the centre of the circle which is to be inscribed in E A C. Then, if another circle be drawn, with L as centre, and L K as radius, it will determine, by the points in which it intersects the diagonals of the square, the centres of the remaining circles.

EXERCISES.

1. Draw a square of 2·75" side, and inscribe another within it, having each of its corners in the sides of the first, and at 1" from its angular points. Draw the circles circumscribing these two squares.
2. Draw a square of 2·75" side. Inscribe in it four equal circles, each touching two others and two sides of the square.
3. Draw an equilateral triangle of 1·5" side; and the four circles, each touching one side of that triangle, and the other two, or those two produced; verify the construction by drawing the circle which would pass through the centres of the three exterior circles.
4. Draw a circle of 1·25" radius, with centre O. The corners of a polygon inscribed in this circle are so placed that the angles at the centre are as follows:—

A O B = 60°	D O E = 80°
B O C = 70°	E O F = 50°
C O D = 50°	

Write down the lengths of A B, B C, and C D.

CHAPTER IV.

ON PROPORTION.

WHEN two numbers or quantities are compared with each other, a ratio is formed. Thus, as 4 : 8 (read as four is to eight) is a ratio. It is readily seen in this instance that the latter number is *twice* the former. We should say, therefore, that 8 : 16 is the same ratio, because 16 is *twice* 8.

Sometimes a ratio is written in a fractional form. Thus $\frac{4}{8}$ is an equivalent expression to 4 : 8.

The first quantity in a ratio is called the *antecedent;* and the second, the *consequent.*

A proportion consists of a number of equal ratios. As 4 : 8 : : 10 : 20 (read as four *is to* eight, *so is* ten *to* twenty) is a proportion, because it consists of the two ratios 4 : 8 and 10 : 20, which are equal to each other. In this case 4 and 10 are the antecedents, and 8 and 20, the consequents.

If the proportion be true, the first antecedent, multiplied by the second consequent, equals the first consequent, multiplied by the second antecedent. This is called multiplying extremes and means.

In all the above instances, abstract numbers only have been used; but concrete quantities are compared in an exactly similar manner. Thus, 4" : 8" is the same ratio as 4 : 8, and 4" : 8" : : 10" : 20" is the same proportion as the one given above. If extremes and means be multiplied in this case, the result will be, that the rectangle made up with 4" and 20" as sides will equal in area another rectangle made up with 8" and 10" as sides.

When quantities are each in an equal ratio with those

which follow them, they are said to be in continued proportion as 4 : 8 : 16 : 32, &c. Here 4 bears the same ratio to 8 that 8 does to 16, and 16 to 32. Then, any pair of alternate terms multiplied together will give the same result as that obtained by multiplying the term between them by itself. Thus, in the above continued proportion, $4 \times 16 = 8 \times 8$.

Problem XXII.

To divide a line A B 3″ *long in the point* C, *so that* A C : B C :: 3 : 5.

Draw the line A B, and at the point A draw a line A C, making *any* angle with A B. Take 3 equal distances of any length, from A to c', and 5 similar distances from c' to b'. Join b' B, and through c', draw a line c' c parallel to b B′. Then c is the required point. The solution of this problem depends upon the principle that in *similar triangles the similar sides are in the same proportion.* It will be readily seen that the triangles A b' B, and A c' c are similar, and that their sides A B and A c are in the same proportion to each other as A b' and A c'.

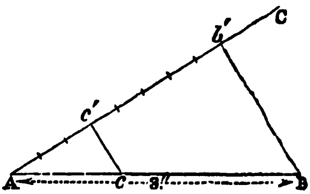


Fig. 27.

The line A B can be divided into any number of equal parts in the same manner, by taking the required number (n) of equal distances along A C, and by drawing a line through each point parallel to n B.

Problem XXIII.

To divide a line A B—3″ *long into* 3 *parts in the points* B *and* C, *so that* A B : A C : A D, *as* 8 : 6 : 5.

At the point A, in A B, draw as before a line, A C, at any angle. Mark off 8 equal distances. At the points 8, 6, 5, numbering from A, place the letters b' c' and d'. Then the line A b is divided in the proportion required. Join Bb', and through c' and d' draw lines parallel to B b', and the line A B will be properly divided in C and D.

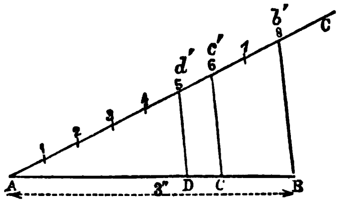


Fig. 28.

Many problems of a similar kind to the two preceding are solved in the same way. For instance, the required point C, in Problem XXII. could be in A B produced, and must be so when the ratio of A C : A B is greater than unity. In that case, the line A c' must be divided as required, and b' joined to B. Then a parallel through c' will meet A B produced, and discover the point C.

Problem XXIV.

To find a fourth proportional to three given lines, A, B, *and* C.

What is required in this problem is the consequent D in the proportion A : B : : C : D.

Take any indefinite line, X Y, and call it a *line of antecedents.* At one extremity make an angle, as X Y Z, and call the line Y Z a *line of consequents.* From the line X Y, cut off a distance, Y 1, equal to the antecedent A, and on Y Z, a distance, Y 2, equal

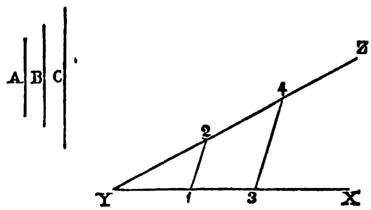


Fig. 29.

to the consequent B. Then, from Y measure the distance Y 3, equal to the given line C. Join 1 2, and through 3 draw 3 4, parallel to 1 2. Then Y 4 is the required line.

It will be clear that there is a fourth proportional smaller than either of these lines. In that case the proportion would read as C : B : : A : D, when C and A would be measured as before; but the point 3 would be joined to 2, and the parallel would be drawn through 1.

Problem XXV.

To find a third proportional to two given straight lines, A *and* B.

A line C is required such, that A : B : : B : C. These three lines will be in continued proportion.

Proceed, as in the preceding problem, by constructing a line of antecedents, X Y, and a line of consequents, Y Z. Then measure on X Y the distance Y 1, equal to the antecedent A, and on Y Z, the distance Y 2, equal to the consequent, B. Describe an arc, 2 3, with centre, Y, and through the point 3 draw a line, 3 4, parallel to 2 1. Then the line Y 4 is the required third proportional.

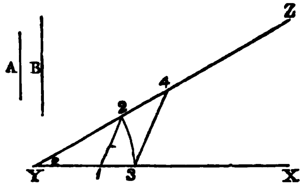


Fig. 30.

Problem XXVI.

To find a mean proportional between two given lines A *and* C.

It will be noticed in the preceding problem that the second and third terms of the proportion are alike. If,

therefore, extremes and means be multiplied, we shall find that A multiplied by C equals B multiplied by itself (B squared); or, in other words, the rectangle made up of A and C is equal in area to a square upon B. The term B of such a proportion is said to be a geometrical mean between the terms A and C.

In the present problem it is required to find this term B, when A and C are given.

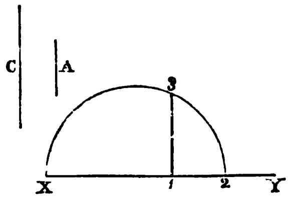


Fig. 31.

Draw a line, X Y, indefinite towards Y, and on it mark off the distances, X 1 and 1 2, equal to A and C respectively. On the line X 2, construct a semicircle, and raise a perpendicular at the point B, to meet the semicircle in 3. Then the distance 1 3 is the required mean proportional B.

Problem XXVII.

To divide a line, A B, *so that the rectangle on the whole line and the lesser segment may equal the square on the greater segment*—(*Euclid,* Bk. II., Def. 11).

At one extremity, A, of the given line raise a perpendicular, A D, equal in length to half the line A B.

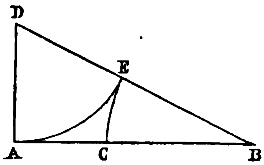


Fig. 32.

Join D B. With D as centre, radius D A, describe the arc A E, cutting D B in E; and with B as centre, radius B E, describe arc E C, cutting A B in C. Then the line A B is divided in the point C, so that a rectangle made up of A B and A C will be equal to the square upon B C.

This is also called dividing a line in extreme and mean proportion, the terms reading thus, as A B : B C : : B C : A C. The greater segment is therefore a mean proportional between the whole line and the lesser.

EXERCISES.

1. A line A B is 2·7″ long. Divide it in the point C, so that A B : B C as 7·5 : 4.
2. Produce a line A B, 3″ long, to a point P, so that B P : A B : : 3 : 5.
3. Two lines, A B and C D, are 3″ and 4″ long respectively. Find a line, E F, so that A B, E F = C D^2.
4. A B is 3″ long, C D 2″. D E = 1·8″. Find a line, F G, such that C D : A B : : F G : D E.
5. A B is the mean proportional between two lines, 3″ and 1·8″ long. Find its length.
6. Divide a 4″ line in extreme and mean proportion, and prove by construction that the greater segment is a mean proportional between the whole line and the lesser segment.

CHAPTER V.

ON THE CONSTRUCTION OF TRIANGLES, POLYGONS, &c.

A triangle is a figure having three sides and three angles; any two sides must be greater than the third side (*Euclid*, Bk. I., Def. 20).

The three angles of a triangle together make two right angles—(180°) (*Euclid*, Bk. I., Def. 32).

The line on which the triangle stands is usually called the base; but any side may, for purposes of practical geometry, be considered as such.

The angle opposite the base is called the vertical angle, and the angular point is the vertex of the triangle.

Note.—The vertical angle of a polygon having an odd number of sides is that angle farthest from the base.

The altitude of a triangle is determined by a perpendicular to the base passing through the vertex.

Note.—When the vertex is not over the base, the latter must be produced.

The perimeter of a triangle is the sum of its sides.

Similar triangles are those having equal angles. A right-angled triangle has one right angle; the lines forming the right angle are called the base and perpendicular, the remaining side being the hypotenuse.

An isosceles triangle has two of its sides equal.

Note.—The angles at the base are also equal (*Euclid*, Bk. I., Def. 5).

PROBLEM XXVIII.

On a line, A B, *2″ long, to erect an Equilateral Triangle.*

Draw a line, A B, 2″ long. With a radius of 2″, and with A and B as centres, draw arcs intersecting in C. Join A C and C B. Then A B C is the required triangle.

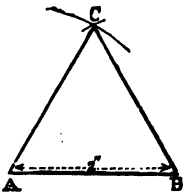


Fig. 33.

PROBLEM XXIX.

A Triangle, A B C, *has its sides* 2″, 1·6″ *and* 2·1″ *respectively. Construct the figure and determine the inscribed circle.*

Draw a line, A B, 2″ long. With A as centre, and with a radius of 1·6″, describe an arc. With B as centre, and with a radius of 2·1″, describe another arc, cutting the former in the point C. Join C A and C B. To determine the inscribed circle, two angles must be bisected, thus :—Mark two points, *e* and *f*, on A C and A B, at equal distances from A. Then, with *e* and *f* as centres, and with equal radii, draw two arcs intersecting in *g*. Join *g* A. Bisect the angle, C B A, in a similar manner. The point where the lines *g* A and *h* B meet is the centre of the inscribed circle.

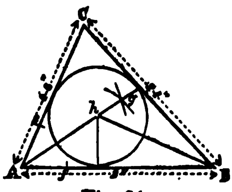


Fig. 34.

Note.—The exact length of the radius may be determined by dropping a perpendicular from the centre on one of the sides, (Prob. III.)

Problem XXX.

To construct an Isosceles Triangle—base 1″, *the vertical angle to contain* 40°.

This problem depends upon the principle that "all the angles in the same segment of a circle are equal." (*Euclid*, Bk. III., Def. 21).

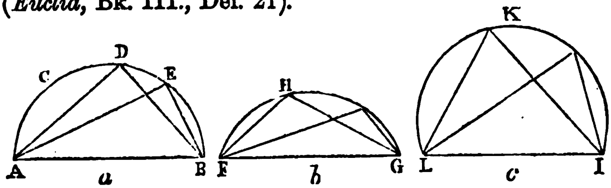


Fig. 35.

Thus the angles A D B, A E B, in the segment A C B (Fig. 35 *a*), are equal.

The same is seen in the segments F H G and L K I (Fig. 35 *b* and *c*).

The angles in a semicircle are right angles, as A E B.

The angles in a segment less than a semicircle are greater than a right angle, as F H G; and the angles in a segment greater than a semicircle are less than a right angle, as L K I.

It remains for us, then, to determine such a segment of a circle as shall contain angles of 40°, and whose chord or base line shall equal the base of the triangle.

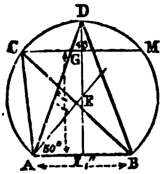


Fig. 36.

Draw the base, A B, 1″ long, and at the point A make A E at an angle of 50° with it. (This angle should always be 90° minus the given vertical angle.) Draw F E perpendicular to and bisecting A B. The intersection of these two lines is the centre of the required segment. As the triangle is to be isosceles,

produce F E, to meet the circle in D. Then D A B is the required triangle.

PROBLEM XXXI.

To construct a Triangle—base 1″, altitude 1·1″, *vertical angle* 40° (See Fig. 36).

This problem is similar to the preceding, in so far as finding the segment A D B. When that is obtained, mark off from the perpendicular, F D, the height (1·1″) required, and draw G C parallel to A B, cutting the circle in C and M. Join C A and C B, and A B C is the required triangle.

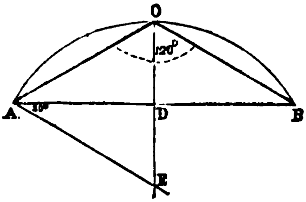


Fig. 37.

There are, therefore, two triangles satisfying the given conditions, one having its vertex in E, and the other in M.

If the vertical angle be greater than 90°, the line A E (Fig. 37), should be made below A B, at an angle equal to that required minus 90°. The remainder of the construction would be as before.

PROBLEM XXXII.

The perimeter of a Triangle is 8″, *its sides are in the proportion of* 1·5, 2·1, *and* 2·8.

This problem is solved by dividing a line 8″ long into three parts, in the given proportion (Prob. XXII.), and then constructing a triangle, having its sides equal to those parts, (Prob. XXIX.)

Problem XXXIII.

To construct a Triangle whose base shall be 2", its perimeter 7", and one of the angles at the base 35°.

Draw the base, A B, 2″ long. At A make A C at an angle of 35° and 5″ long. These two lines will together

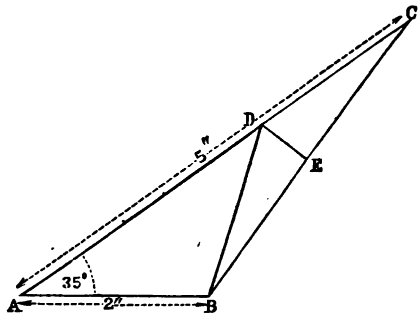


Fig. 38.

equal the perimeter. Join B C, and at the centre E, draw E D perpendicular to B C, cutting A C in D. Join D B; and A D B is the required triangle.

Problem XXXIV.

The angles of a Triangle being as 2 : 4 : 3, and the base 2″; to construct it.

Draw a line, A B, 2″ long. Produce it beyond A and, with A as centre, describe a semicircle (radius at pleasure). Divide the semicircle, with the dividers, into 9 $(2+3+4)$ equal parts, and draw lines from the centre, A, to the points 2 and 6. The angles 2 A D, 2 A 6. and 6 A 9 will then be in the required propor-

tion; and as they together make two right angles, they must be the angles of the required triangle. Through B, therefore, draw B C parallel to 2 A, till it meets A 6 produced in C; then A B C is the required triangle.

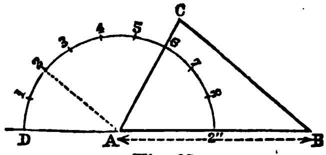


Fig. 39.

PROBLEM XXXV.

The perimeter of a Triangle is 7", its angles are as 2 : 4 : 3. Construct it, and add the circumscribing circle.

The angles are determined by the construction explained in the preceding problem. A triangle, A B C, having angles equal to those found, must then be constructed. (The base of this figure may be assumed as of any length.)

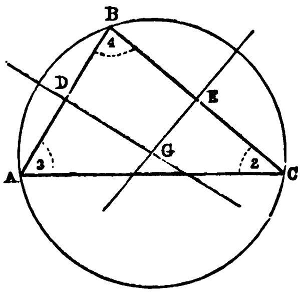


Fig. 40.

The length of the sides of the required triangle can be obtained by dividing a line 7" long (the given perimeter) into three unequal parts, in the proportion of A C : B C : A B, (Prob. XXII.) Construct then a triangle, with these three segments for sides, and the figure will be determined as required.

To circumscribe the triangle with a circle, find the centre of two of the sides, as D and E, and draw lines,

D G and E G, perpendicular to them, intersecting in G. This is the centre of the required circle.

Problem XXXVI.

To construct a Right-angled Triangle whose base shall be 2" long and the hypotenuse 4".

Draw the base, A B, 2" long, and at the point A raise a perpendicular, indefinite in length. With B as centre, radius 4", cut this perpendicular in C. Join C B, and A B C is the required triangle. If the hypotenuse be bisected in D, and an arc be drawn passing through the points A and B, it will also pass through the point C, because, as we have already learned, the angle in a semicircle is a right angle.

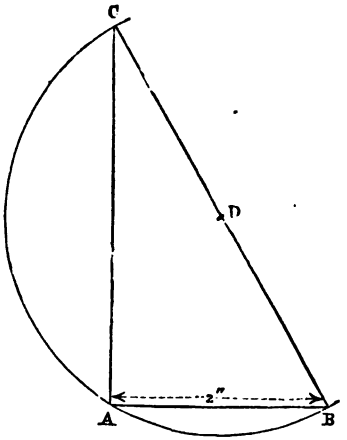


Fig. 41.

Problem XXXVII.

To construct a Right-angled Triangle whose base, A B, *shall be 2" and its angle,* A C B, 38°.

As the three angles of every triangle together

make two right-angles, and as in the required triangle one angle is to be a right angle, the remaining two must together make 90°. The angle, A C B, given in the problem is that opposite to the base; consequently, the other acute angle is equal in magnitude to 90° − 38° (52°). At the point B, therefore, set off this angle. The line B C will then meet an indefinite perpendicular through A, in the vertex, C, of the required triangle.

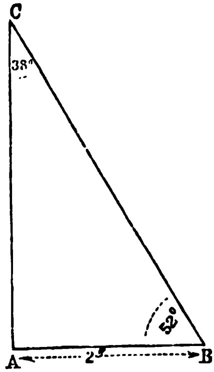


Fig. 42.

Problem XXXVIII.

To construct a Right-angled Triangle whose hypotenuse shall be 3″ long and one of the acute angles 35°.

As a semicircle holds a right angle, bisect the given line B C, which is to be the hypotenuse, and describe a semicircle upon it. At one extremity of the diameter draw a line, making an angle of 35° with it. This line will intersect the semicircle in the point A, which is the third corner of the required triangle.

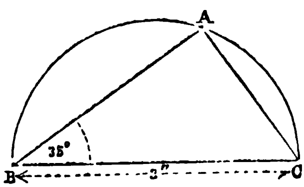


Fig. 43.

Problem XXXIX.

To construct a Right-angled Triangle, the base, A B, *and the perpendicular,* A C, *to be in the proportion of* 3 : 4, *and the hypotenuse,* B C, 3″ *long.*

Draw two lines, A 4, A 3, perpendicular to each other. From the angular point A mark off three equal distances (any length) along A 3, and four similar distances along A 4. Join 3 to 4, and produce the line 3 4, making 3 D, 3″ long. Through the point D draw a line, D C, parallel to the base, and meeting the perpendicular in C.

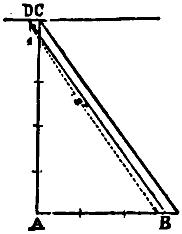


Fig. 44.

Through C draw B C parallel to 3 4, meeting the base line in B. Then A B C is the triangle required.

Problem XL.

To construct a Triangle whose perimeter shall be 6″, *altitude* 1·7″, *and one of the angles at the base* 42°.

As the altitude of the triangle is to be 1·7″, it is

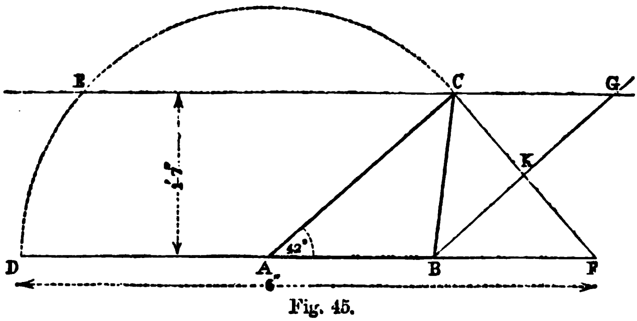


Fig. 45.

evident that the figure will stand between two parallels, D F and E C, that distance apart. Draw these, therefore, and at any point, A, in one of them make the angle F A C – 42°. One side, A C, of the figure is now determined. From A set off the distance A F, so that A C, together with A F, shall equal 6″. Join F C, and at its centre draw the perpendicular K B. Join B C, and the triangle will be completed.

Problem XLI.

To construct a Triangle whose vertical angle, A C B, *shall be* 36°, *the sides,* A C *and* B C, *in the proportion of* 3 : 4, *and the base,* A B, 1·5″ *long.*

At any point, X, make two lines, F X and D X, meeting at an angle equal to the given vertical angle. On F X, mark off three equal distances measuring from X (length at pleasure); and on D X, four of such distances. Join the points found, and cut off A B, equal to 1·5″. Through B draw B C parallel to E X, and the required triangle will be determined.

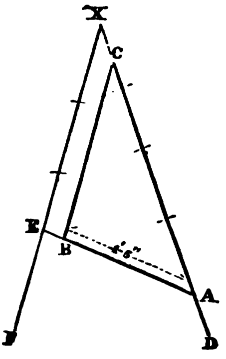


Fig. 46.

Polygons.

For definitions, &c., of Polygons, see Chapter II.

All the angles of a regular polygon are equal, and if they be added together their sum will equal twice as many right angles as the polygon has sides, less four. (*Euclid*, Bk. I., Def. 32.)

Thus, in an octagon, the sum of the interior angles will

be $(16 \times 90^\circ) - 360^\circ = 1{,}080^\circ$. And as they are equal, each angle must be $\frac{1080^\circ}{8} = 135^\circ$.

Lines which bisect and are perpendicular to the sides of a regular polygon meet in one point—the centre.

Lines drawn from the angular points of a regular polygon to the centre divide the figure into a number of isosceles triangles.

A circle can be drawn to pass through all the angular points of a regular polygon. This is called the circumscribing circle.

A circle can also be drawn to which the sides of a polygon shall be tangent. This is the inscribed circle.

Both these circles have the same centre as the regular polygon.

PROBLEM XLII.

A method of constructing either of the regular Polygons upon a given line, A B.

Let A B be the given line, and a pentagon the required polygon. Produce A B to F, making F A equal to A B. With centre, A, draw the semicircle, F E B, and divide it by trial into five equal parts. (The number of these divisions must correspond with the number of the sides of the polygon required.) Join A E—E being always the *second* division from the extremity of the semicircle. Then A B and A E will be two sides of the figure. Raise perpendiculars at the centres of these lines, meeting in O. With O as centre—radius O A—describe the circle, A B C D E, and mark off distances,

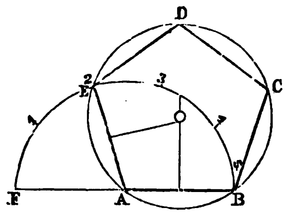


Fig. 47.

B C and C D, equal to A B. Join B C, C D, and D E, and the figure will be complete.

PROBLEM XLIII.

A method of inscribing any regular Polygon in a given circle.

Let A C B be the given circle, and let the required figure be a heptagon. Draw the diameter A B, and divide it into seven equal parts. (The number of parts is regulated by the required number of sides.) With A and B as centres—radius A B—describe two arcs intersecting in D. From the point D draw the line D2, passing through the second division of the diameter, and produce it, to meet the circle in E. The distance, A E, will divide the circle into seven equal parts; and if the points of division be joined, a heptagon will be inscribed in the circle.

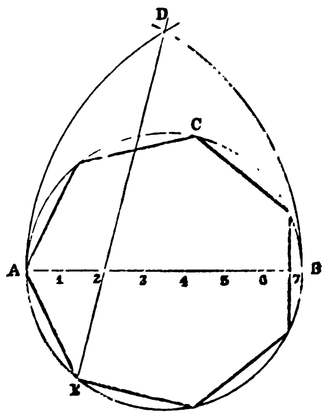


Fig. 48.

In the case of the hexagon, the radius of the circle will give the length of the side.

For an octagon, draw two diameters perpendicular to each other, and bisect the quadrants.

There are special methods for determining each of the regular polygons; but it is not necessary for the student to consider these if the two preceding problems be understood.

Problem XLIV.

To construct a Pentagon whose diagonal shall be 3" long.

Draw a line, P Q, and mark any point, A, in

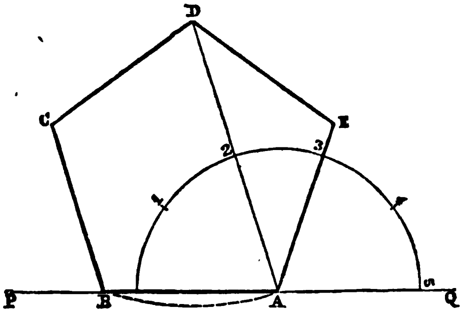


Fig. 49.

that line. With A as centre, describe a semicircle. Divide this semicircle into five equal parts, and draw the line A D, passing through the second division. Make A D 3" long, and with D as centre, radius 3", cut the line P Q in the point B. Then A B is the base of the pentagon. To complete the figure, draw arcs with A, D, and B as centres—radius, A B intersecting in E and C.

Join B C, C D, D E, and E A. Then A B C D E will be the required pentagon.

EXERCISES.

1. Construct a triangle, its sides 2" and 3", and the included angle 50°.
2. Draw an equilateral triangle, whose perimeter shall be equal to a square of 1" side.
3. Construct a triangle from either of the following conditions:—

Its sides as 3 : 4 : 5, and its perimeter 13".
Its base 5", altitude 2·5", and its vertical angle 85°.

4. Construct a triangle whose perimeter shall be 7"; base, A B, 2"; and angle, B A C = 50°.
5. Construct a pentagon having its side 2".
6. Place two equal lines, 1·5" long, at an angle of 135°. Consider them as two sides of a polygon, and complete the figure.
7. Construct a triangle, two of whose sides are 2·5" and 3·25", the angle opposite the shorter one being 40°. Draw also the circumscribing circle.
8. Construct a triangle, A B C, having its angles 50°, 60°, 70° and circumscribing a circle of 1" radius.
9. Construct a right-angled triangle whose base is half the length of its perpendicular, the hypotenuse being 4".
10. Construct a right-angled triangle, base 1", the acute angles being in the proportion of 2 : 1.
11. In a circle inscribe a heptagon.
12. Draw a pentagon, side 2", divide it into five equal isosceles triangles, by drawing lines from its centre to the angular points, and inscribe a circle in each.

CHAPTER VI.

ON THE AREAS OF PLANE FIGURES.

The area of a plane figure is the amount of surface enclosed by its boundary line, called "the perimeter," and, although depending upon the position and length of the lines forming that perimeter, is not measured by the same standard. This is easily seen by cutting a piece of paper square, each side being 1" long. The edges of it together measure 4" in length; but the area is what is termed *one square inch*. Again, if we cut another piece of paper of the same shape, but each side 2" in length, it is evident that it will be four times as large as the first piece, and contain therefore four square inches. The edges of the latter piece of paper will measure 8"; and thus we see that with only twice the perimeter we get four times the area.

The boundary line, therefore, of a figure does not alone determine the amount of its surface.

Shape plays a great part in settling the area of plane figures, and certain relations existing between them are demonstrated in works on Theoretical Geometry. A few of these are given, which the writer would advise all students to thoroughly comprehend before attempting the problems of this chapter.

a. The squares on the base and perpendicular of a right-angled triangle are together equal in area to the square on the hypotenuse (*Euclid*, Bk. I., Def. 47).

b. Parallelograms (squares, rectangles, and rhomboid figures) or triangles upon the same or equal bases, and between the same parallels, are equal in area (*Euclid*, Bk. I., Defs. 35-38).

c. If a parallelogram and a triangle be upon the same

base, and between the same parallels, the former will be double of the latter in area (*Euclid*, Bk. I., Def. 41).

d. The area of a triangle is measured by a rectangle, having the altitude and half the base as sides.

Note.—The rectangle on the base and half the altitude gives the same area.

e. The areas of similar shaped figures are in the same proportion as the squares on the similar sides, (*Euclid*, Bk. VI.)

f. Circles are in the same proportion, as to area, as the squares on their diameters, (*Euclid*, Bk. VI.)

g. Perimeters being equal, the greatest space is enclosed by figures having equal sides, and the greater the number of sides the greater the area.

Note.—A circle, therefore, holds the greatest area with the shortest boundary line, called in this case the *periphery.*

PROBLEM XLV.

Given any irregular Triangle, A B C, *to construct an Isosceles Triangle upon the same base, equal to it in area.*

Referring to the commencement of the chapter, we find that triangles upon the same base and between the same parallels are equal.

It is clear, then, that if we draw a line, *e f*, through the apex C, of the given triangle, parallel to its base A B, that line will be the *locus** of the apices of all triangles on the same base, which are equal to the given one. If we bisect the base and raise a perpendicular *g* D, to meet

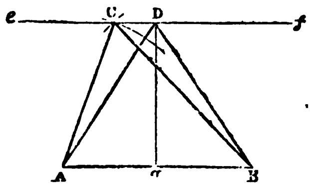


Fig. 50.

* If a line be determined in which, under certain conditions, a point must exist, that line is a *locus* of the point.

this parallel in D, we shall obtain the apex of the required figure, and can then complete it by joining this point to the extremities of the base.

PROBLEM XLVI.

To make a Rectangle equal to a given Triangle, A B C.

If the two figures are to stand upon the same base, the rectangle must be half the height of the triangle; but if the height is to be the same, the rectangle must stand on half the base.

PROBLEM XLVII.

To construct an Isosceles Triangle equal in area to a given Square.

The triangle will have its apex over the middle of th base of the square, and will be twice the height of it.

PROBLEM XLVIII.

A Rectangle 2" by 1" being given, required a Square equal to it in area.

A B F E is the rectangle. Produce A B beyond B, making B G equal to B F. Find a mean proportional to A B and B G, (Prob. XXVI.) Then B C, the mean proportional found, is the side of the required square.

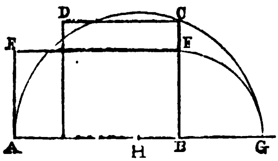


Fig. 51.

Problem XLIX.

Given one side of a Rectangle 2·5", to complete the figure, so that it may be equal in area to a square of 1·2" side.

This is the converse of the last problem. Set the given sides of the rectangle and square perpendicular to each other, as A B, B C. Join A C, and at its centre, D, set out the perpendicular, D E, cutting A B in E. With E as centre, describe an arc passing through C, and meeting A B, produced in the point F. Then B F is the second side of the rectangle, and the figure can be completed.

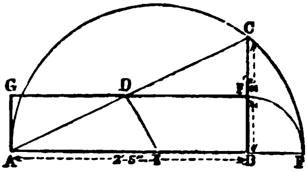


Fig. 52.

Problem L.

To construct a Triangle with sides 2", 1·7", and 1·2" respectively, and a Square equal to it in area.

Construct the triangle (Problem XXIX.), and make a rectangle equal to it in area, (Problem XLVI.) Convert this rectangle into a square, (Problem XLVIII.)

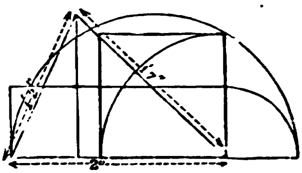


Fig. 53.

PROBLEM LI.

To make a Square equal to two given Squares of ·8" and ·5" sides.

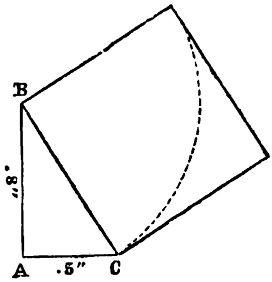


Fig. 54.

Upon two lines perpendicular to each other mark off the given lengths, as A B (·8") and A C (·5"). Join B C, and upon it construct the required square. This problem depends upon the principle *a* mentioned at the commencement of this chapter.

By similar means we are enabled to make a square of any given area, say 3 square inches. We place two lines, each 1" long, perpendicular to each other, as A B, B C (Fig. 55), and join A C. This line is the side of a square of 2 square inches area. We then make C D 1" long perpendicular to A C, and join A D. Then A D is the side of a square 3 square inches in area.

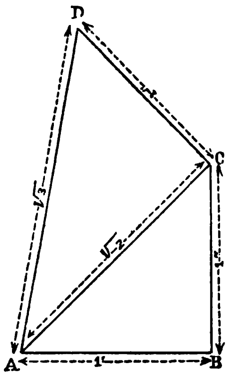


Fig. 55.

If lines 2" and 1" long be used as the base and perpendicular of a right-angled triangle, the hypotenuse will be the side of a square 5 square inches in area. It is usual to consider the lines as the square roots of the units of area enclosed in their respective squares, and to write them thus, $\sqrt{3}$, $\sqrt{5}$, &c., (read as the square root of 5, square root of 3, &c.)

If a semicircle be made upon a line 1" long, and the

curve be bisected, the line joining the centre of the semicircle to either of its extremities is the side of a square, which will contain *half* a square inch, or in other words, represents the root of $\frac{1}{2}$ (unit 1″).

The principle *a* is involved in this last construction, as the angle made by the lines joining the centre of the semicircle to the extremities of its diameter is a right-angle. The diameter plays the part of the hypotenuse.

PROBLEM LII.

To determine a Circle equal in area to that of two given Circles added together.

The solution of this problem is similar to that of the preceding one, as the diameters of circles can be treated as regards area in the same manner as sides of squares. Make, therefore, a right-angled triangle whose base and perpendicular are respectively equal to the diameters of the given circles. Then half the hypotenuse is the radius of the required circle, equal in area to the sum of those given. (See Principle *f*.)

PROBLEM LIII.

Having given two similar Polygons (regular or irregular): to construct a third Polygon, alike in shape but equal in area to that of the former two added together.

This is merely a modification of the two preceding problems, as we see, by principle *e*, that the sides of similar polygons can be treated as regards area like those of squares or the diameters of circles. Place, therefore, a similar side of each of the two given figures perpendicular to each other, and assume the line which joins the opposite extremities of these, as the corresponding side in the required figure.

Note.—A square, circle, or polygon, can be found equal to more

than two given squares, circles, or similar polygons. For, if the sum of the first two be added to the third, and the sum of these again to the fourth, &c., the result will be that a figure will be determined equal in area to the whole.

Problem LIV.

To determine the side of a Square of 3·5 square inches area.

This problem can be solved in two ways. We can by the method explained in Problem LI. find the root of 3 square inches, and at the extremity of this line raise a perpendicular equal in length to the root of one-half a square inch. We can then make the hypotenuse upon these two lines, and it will be the side of a square of 3·5 square inches area.

The other method of working this problem is as follows:—Make a rectangle of 3·5 square inches area, one side of which it would be most convenient to assume as 1″ long, the other being 3·5″ long. Find a mean proportional between the sides of the rectangle, and it will be the line required.

It will be seen that, by this latter method, the sides of any square can be found containing fractional parts of a square inch; for if 1″ be used as one side of the rectangle, the other side must always be the same measure of inches in length as the required square contains in area.

Problem LV.

To make a Square equal in area to the difference between two given Squares.

Let A B and C D represent the sides of the given squares. At B raise a perpendicular, B E. With A as centre, radius equal to C D, describe an arc, intersecting the perpendicular in F. Then B F is the side of the required square.

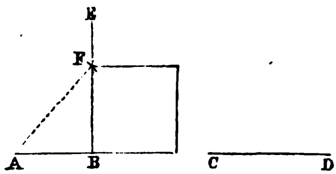


Fig. 56.

This is easily proved; for as the squares upon the base and perpendicular of a right-angled triangle are together equal to the square upon the hypotenuse, therefore the square on the perpendicular must be equal to the difference of those upon the hypotenuse and base.

A circle equal in area to the difference between two given circles can be determined in an exactly similar manner by treating the respective diameters as sides of squares.

Problem LVI.

Given any Triangle, to make a similar one of double the area.

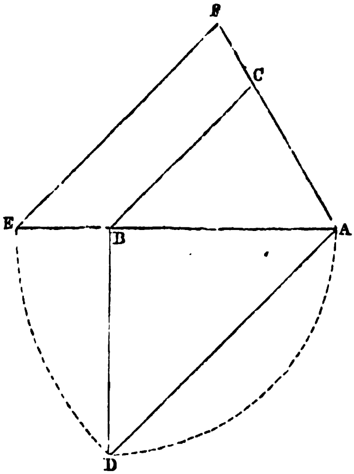


Fig. 57.

Let A B C be the given triangle. At one extremity of the base, as B, drop a perpendicular, B D, equal in length to A B. Join A D, and with A as centre, A D as radius, describe the arc, D E, meeting A B produced in E.

Then A E is the base of the required triangle, which can be completed by drawing through E the line E F parallel to B C, intersecting A C produced in F.

Here, again, the principle *a* is involved, but it is combined with the principle *e*.

Note.—The above construction will enable the student to make a triangle three or any number of times larger than the given one. For if he places a line, equal to A B, perpendicular to

another line, equal to A E, the hypotenuse upon these two will be the base of a triangle three times the area of the given one.

Problems upon this principle could be multiplied indefinitely; but no intelligent student could feel any difficulty in solving them, if he understands the geometrical truths involved in the last few cases.

Problem LVII.

To reduce any irregular figure to a Triangle of equal area.

It is advisable, in working this problem for the first time, to assume a figure having no more than five

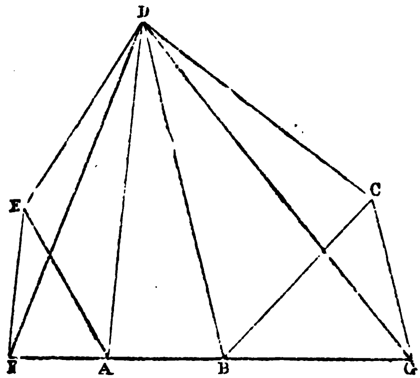


Fig. 58.

sides, and when the principle is understood, to apply it to more complicated cases.

Let A B C D E be the polygon. Produce the base, A B, in both directions. Join A D, and notice, in doing this we cut off a triangular piece, A E D, of the figure. Through E draw E F parallel to A D, and join D F.

Then, by principle *b*, it can be shown that the triangles, A D E and A D F, are equal; for they are both on the same base, A D, and have their vertices in the parallel, E F. But the triangle, A D F, has one of its sides in the line A B produced, and the figure F D C B is equal to the given polygon, but has only four sides. Hence their number has been reduced by one.

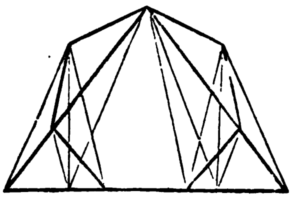


Fig. 59.

Joining B D, and proceeding as before, we lessen the number of sides again to three, and so obtain a triangle, F D G, equal in area to the original figure.

Note.—If the given polygon has many sides, proceed as before to reduce them, one at a time, till the required triangle is obtained, always commencing *from* the line to be used as base, on either side. The case of a heptagon reduced to an equal triangle is shown in Fig. 59, which the student will, by a little consideration, and by thoroughly comprehending the principles involved in the last case, be able to understand for himself without difficulty.

Problem LVIII.

To make an Isosceles Triangle with a vertical angle of 40° *the area of which shall be* 3·5" *square inches.*

Make an angle of 40°, as at C, and cut off equal distances, C A and C B, on each of the legs of the angle. Join A to B, and a triangle similar in shape to the one required in the problem will be obtained. A square equal in area to this figure can be deduced (Prob. L.), by making a rectangle first upon the same base, A B, and half the height of the triangle, and finding a mean proportional between the sides of the rectangle.

Produce the perpendicular, B F, and mark off B H

equal to $\sqrt{3 \cdot 5}$, (Prob. LIV.) Then the two lengths, B E and B H, will represent the sides of two squares equal to the assumed and required triangles respectively. Join F C, and draw H K parallel to it, meeting B C produced in K.

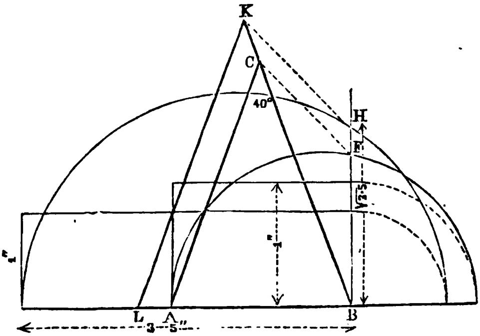


Fig. 60.

Draw K L parallel to A C, meeting A B produced in L, and B K L is the required triangle.

This problem depends upon another very useful principle—that of assuming a figure of wrong area, and deducing from it another of a required size.

Any figures can be compared in this way by finding the sides of squares equal to them in area, as similar figures are to one another as the squares upon their similar sides.

Problem LIX.

To construct a Rhombus, having an Angle of 60° *of* 5 *Square Inches area.*

Make any rhombus, A B C D, with a given angle of 60°. A mean proportional between the base A B and the altitude B E of the figure will give the side B F of

a square equal to the assumed rhombus. Produce B F, and make B G equal to $\sqrt{5 \text{ square inches}}$. Join F C, and

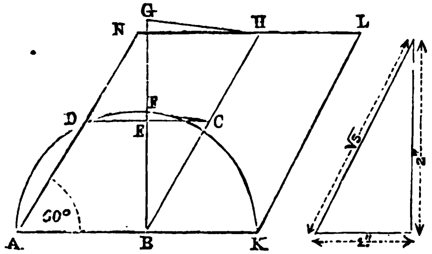


Fig. 61.

draw G H parallel to it, meeting B C produced in H. Then B H is equal to the side of the required rhombus.

Problem LX.

To make a Triangle equal to the given Triangle A B C, *and having* A D *for its base.*

Join D C, and through the point B draw B E parallel to C D, join E D, and the triangle A E D will be equal to the given one, A B C.

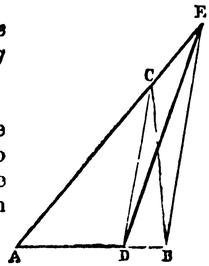


Fig. 62.

Problem LXI.

To make an Equilateral Triangle equal to any Irregular Triangle A B C.

Make an equilateral triangle on one of the sides as

A D C. Produce either of the sides D C, D A. Through B draw B E parallel to A C, and meeting D E in E. Then a mean proportional C F between D C and C E is the side of the required equilateral triangle C F G.

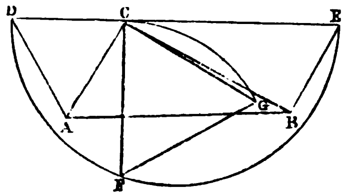


Fig. 63.

Problem LXII.

To make an Equilateral Triangle of 5 Square Inches area.

This problem can be solved by an exactly similar method to that adopted in Problem LIX., by assuming an equilateral triangle on any base, and deducing therefrom, one of the required area. It can also be solved by finding a square of 5 square inches area, and making an isosceles triangle equal to this square, afterwards deducing an equilateral triangle of the same area by the construction of the preceding problem.

Problem LXIII.

To make an Equilateral Triangle equal to the irregular Polygon (fig. 58).

This needs no detailed description. It is solved by Problems LVII and LXI.

Problem LXIV.

To divide a given Triangle A B C, *into any number* (4) *of equal parts by Lines parallel to the side* A B.

Divide one of the sides of the triangle, as B C, into

the required number of parts. Upon B C describe a semicircle, and raise perpendiculars from the points 1′, 2′, and 3′, meeting the semicircle in 1, 2, and 3. With C as centre, draw arcs passing through 1, 2, and 3, intersecting B C in D, E and F.

Then parallels to the base drawn through these points will divide the triangle as required.

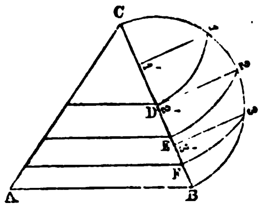


Fig. 64.

Problem LXV.

To bisect the Triangle A B C *by a Line perpendicular to the base* A B.

Bisect the base in the point D, and draw a line E C perpendicular to it. Find a mean proportional between either of the unequal segments A E, E B, and half the base.

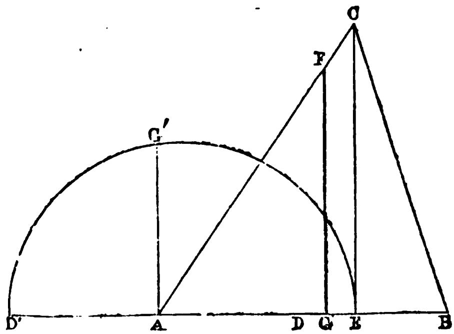


Fig. 65.

In the accompanying figure, A G′ is the mean proportional between A D and A E (A D′ = A D).

Measure the distance A G equal to A G′ from A along the base to G. At G raise the perpendicular G F. Then the triangle will be bisected as required.

NOTE.—The length of the mean proportional must be measured from the same extremity of the base, as the segment used for the mean proportional.

PROBLEM LXVI.

To divide the Triangle A B C *into any number of equal parts by Lines drawn through the apex.*

This is solved by dividing the base into the required number of parts and joining the points of division to the apex. The triangles thus formed are equal because they are on equal bases and of the same altitude.

PROBLEM LXVII.

To divide the Triangle A B C *into any number* (4) *of equal parts by Lines drawn through a point* P *in one of the sides.*

Divide one of the sides in which the given point does not occur into the given number of parts as 1, 2, 3.

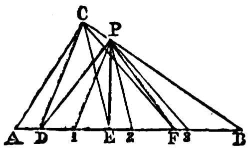


Fig. 66.

Join 1 to P, and through C draw C D parallel to P 1, intersecting A B in D. Join P D. Then P C A D is one fourth of the whole triangle A B C.

Join 2 and 3 to P, and draw C E and C F parallel to 2 P and P 3 respectively, intersecting the base in E and F. Join P E and P F, and the problem will be solved.

PROBLEM LXVIII.

To bisect any irregular figure by a Line drawn from one of its corners.

Let A B C D be the given figure and A the given corner. Draw the diagonals A C and B D. Bisect the

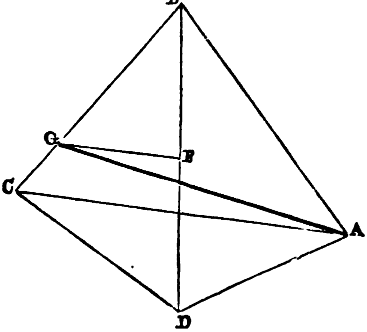


Fig. 67.

diagonal B D in F, and through the point F draw FG parallel to A C, meeting B C in G. Join A G, and the figure is bisected.

PROBLEM LXIX.

To divide the Rectangle A B C *into any number* (3) *of equal parts, by Lines drawn through the given point* P.

Divide the side in which the given point is situated into three equal parts, and draw perpendiculars from each of the points of division. Bisect these perpendiculars and draw lines from the given point P, through each of the points of bisection.

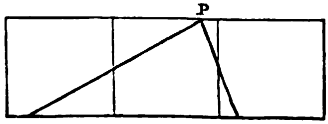


Fig. 68.

Note.—When the rectangle is to be divided into more than three parts, some of the dividing lines may not meet the base until it is produced. When such is the case the solution is correct

for all those parts which have their bases in that of the rectangle, but incorrect for the others. Generally the remaining space has to be divided into two equal parts by the preceding problem. The student need not at this stage consider a more complex case.

Problem LXX.

To divide a Square into any number (5) of equal parts, by Lines drawn through one of its corners.

Let A B C D be the required square. Divide the sides, B C and C D, each into five equal parts in the points 1, 2, 3, 4, &c. Join 2, 4, 5 and 7 to the corner A, and the figure will be divided as required.

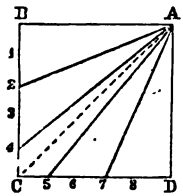


Fig. 69.

The triangles formed are of the same area, because their bases, and altitudes are equal. The quadrilateral figure, A 4 C 5, it will be seen, is made up of two smaller triangles, each equal to a half of either of the larger ones.

Problem LXXI.

To divide a Square into any number (5) of equal parts by Lines parallel to the diagonal.

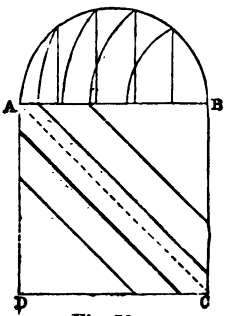


Fig. 70.

Let A B C D be the square, and A C its diagonal. Divide the triangles A B C and A D C each into five equal parts, by the construction explained in Problem LXIV. The square will then be divided into ten parts. By taking alternate divisions, as shown in fig. 70, and rubbing out the diagonal, the figure will be in five parts, as required by the problem.

Problem LXXII.

To convert a Rhomboid figure into a Rhombus having an equal angle.

Treat this problem in exactly the same way as that of converting a rectangle into a square; that is, find a mean proportional between the unequal sides of the rhomboid figure, and make a rhombus with the same included angle, its sides being equal to the line determined.

Problem LXXIII.

To construct a Rectangle of 3 *square inches area, its sides to be in the proportion of* 3 : 2.

Upon an indefinite line, A C, mark off two distances, A B, B C, which shall be in the proportion of 3 to 2.

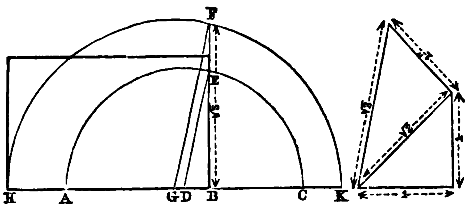


Fig. 71.

Find a mean proportional, B E, between them. Join E to D. Mark off B F upon B E produced, equal to $\sqrt{5}$ square inches, and draw F G parallel to E D. With G as centre, radius G F, describe the semicircle H F K, cutting the line A B in points H and K. Then a rectangle, with H B and K B—which are as 3 : 2—as sides, will contain 5 square inches.

Problem LXXIV.

To construct a Square equal in area to any irregular quadrilateral figure.

Let A B C D be the given figure. Draw its diagonal, A C, and the lines B E and D F, perpendicular to A C. Bisect B E and F D in G and H. Find a mean proportional between the diagonal A C and the sum of the two lines, E G and F H. Then C K is the side of the required square.

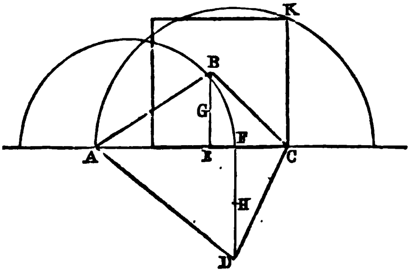


Fig. 72.

Problem LXXV.

To construct an Isosceles Triangle equal to a Regular Pentagon (base 1").

If lines be drawn from each of the angular points of the polygon to its centre, the figure will be divided into five equal triangles. To solve the above problem we must either make an isosceles triangle, with a base equal to that of the pentagon, and with a height five times that of one of the small triangles, *or*, the height being the same, the base must be five times as long.

By means of this construction a square, equilateral triangle, &c., can be easily determined, equal to any given polygon.

EXERCISES.

1. Construct a triangle, perimeter 7″, area greater than that of any triangle of equal perimeter.

2. Determine an equilateral triangle, equal in area to the sum of 2 squares of 1 and 2 inches side.

3. Construct a square of 5 square inches area.

4. Construct a triangle—its sides as 7 : 8 : 9—area, 3 square inches.

5. Bisect a triangle, having its sides 3·5, 4, 4·5 inches by a line, either—

   a. Parallel to the shortest side;
   b. Or perpendicular to the longest;
   c. Or by a line drawn from a point at 1″ from either end of the longest side.

6. Draw an equilateral triangle of 1·5″ side, and a square equal to it in area.

7. Make any irregular figure of four or six sides, and construct an equilateral triangle equal in area.

8. Draw two circles equal to the sum and difference respectively of two other circles of 1·8″ and 3″ diameter.

9. Make a pentagon of 5 square inches area, and a second one twice as large as the first.

10. Construct a square, an equilateral triangle, and a hexagon. Determine by a square the area of the three figures added together.

11. Find by construction the value of the following expressions—the unit being 1 square inch.

$$\sqrt{2} \qquad \sqrt{5} \qquad \sqrt{3} \qquad \text{and} \qquad \sqrt{3\tfrac{3}{4}}$$

12. Construct a pentagon having a diagonal 3″ long, and a square equal to it in area.

CHAPTER VII.

PROBLEMS ON THE LINE AND CIRCLE.

Facts to be remembered.—A tangent to a circle is perpendicular to the radius drawn through the point of contact. When two circles are tangential, the line joining their centres passes through the point of contact (*Euclid*, Bk. III., Def. 12). Only two tangents can be drawn from one point to the same circle, and these are equal to each other.

PROBLEM LXXVI.

In the given Angle A, *to inscribe a circle of* 1·2" *radius.*

Bisect the angle A by the line A D, and draw E F parallel to A B at a distance 1·2" from it. This line will intersect A D in the point E. Describe the required circle, with E as centre, radius 1·2". The points of contact, G and H, of the circle and legs of the angle can be determined by drawing the radii E G and E H perpendicular to A B and A C respectively.

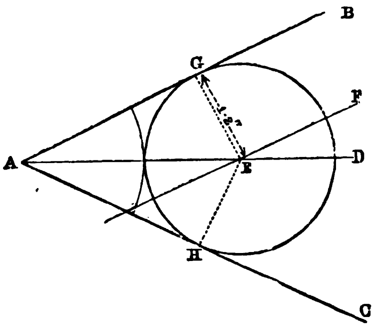


Fig. 73.

PROBLEM LXXVII.

To inscribe a succession of Circles in a given angle, each circle to be tangent to those which precede and follow it.

Bisect the angle, and with any point, E, as centre, describe a circle touching the lines A B, A C. Show the point of contact, H, as above, and at G, draw a line tangent to the circle (that is, perpendicular to A D) meeting A B in K. If we consider for a moment that K G and K B must both be tangent to the next circle,

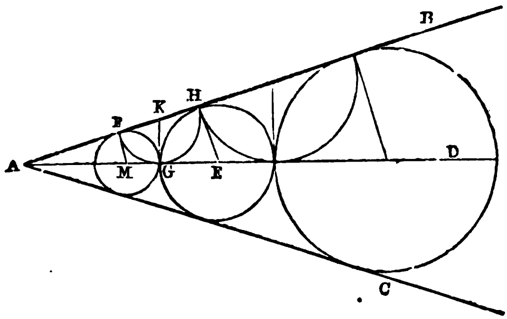


Fig. 74.

we shall see that by striking an arc, with K as centre and K G as radius, we shall obtain F, which is the point of contact of A B with that circle. F M must then be drawn perpendicular to A B, to determine the centre, M.

Describe the circle with M as centre and F M as radius. By similar construction, the centres of a series of circles can be found as required by the problem.

PROBLEM LXXVIII.

To draw two Lines tangent to a given circle C, *meeting at any given angle* (57°).

From the centre, C, draw two radii, making an angle with each other equal to the supplement of the given one (180°–57°).

At the extremities of these radii, draw the required tangents, perpendicular to them.

Problem LXXIX.

To inscribe a Circle in a given angle which shall pass through a given point C.

Bisect the given angle, and with E as centre describe a circle touching the lines A B, A D. Join C to the point A, intersecting the circle in G. Draw the radius G E, and a line C K parallel to G E will determine the centre, K, of the required circle, the radius being K C.

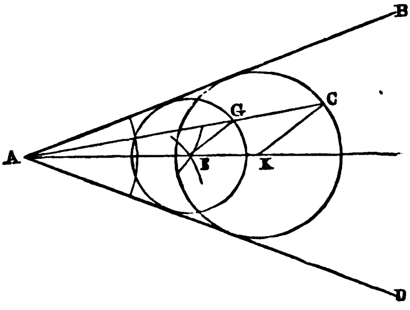


Fig. 75.

Problem LXXX.

A Circle of ·75" *radius, has its centre* 1·5" *from a straight line* X Y: *required, a circle of* ·5" *radius to touch both the given line and circle.*

Through the centre, A, of the given circle draw a line, A B, perpendicular to the given line X Y. Make C F parallel to X Y at a distance from it equal to the given radius (·5"). With A as centre, radius ·5" greater than that of the given circle, describe the arc E F meeting the parallel, C F, in F. Then F is the centre of the required circle.

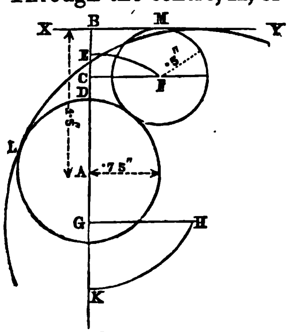


Fig. 76.

Note.—The line C F is the

locus of the centres of all circles with radius ·5″, to which X Y would be tangent.

Similarly, the circle E F is the locus of the centres of all circles, of ·5″ radius, which would be tangent to the given circle, A.

The large circle passing through L is 2″ in radius and includes the given one, the construction being a modification of the above.

Problem LXXXI.

Two Circles of 1″ and ·5″ radius respectively have their centres 2″ apart; required another circle of ·75″ radius to touch both the former externally.

Draw a straight line and mark off a distance, A B, of 2″. Describe the circles A and B, with their respective radii, as given in the question. To determine the centre

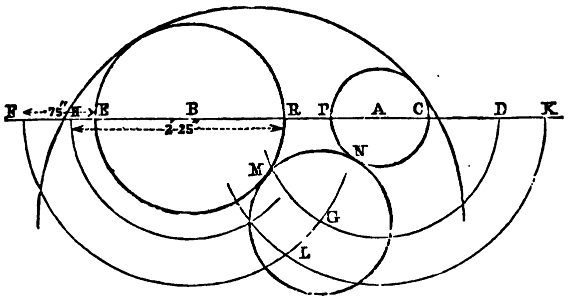


Fig. 77.

of the third circle, produce A B to F, making C D and E F each equal to the given radius, ·75″. With A and B as centres, describe the arcs D G and F G intersecting in G. Then G is the centre of the required circle.

The points of contact, M and N are determined by joining the centres of the circles.

Problem LXXXII.

Given the Circles as in Problem LXXXI.; to describe a Circle of 2·25″ radius which shall be tangent to both and include them.

The construction of this problem is shown in fig. 77. On the line A B, make R H and P K equal to the given radius (2·25″). With A as centre, describe the arc K L, and with B as centre describe the arc H L. These arcs will intersect in the point L, which will be the centre of the required circle.

In all cases where the required circle is to be external to the given one, the radius of the former should be added to that of the latter, but when the given circle is to be included, the radius of the required one should be measured across its diameter.

Problem LXXXIII.

A Line, A B, *is* 1·5″ *from the centre of a circle of* ·8″ *radius. A point,* P, *in the line is* 2″ *from the centre of the circle. Draw a second Circle to touch the line* A B *in the point* P, *and also the given circle* (1.) *externally,* (2.) *and to include it.*

(1.) Through the given point P draw an indefinite line, K L, perpendicular to A B. Mark off the distance P D equal to the radius of the given circle. Join C D, and bisect in the point E. Draw E F perpendicular to C D and intersecting K L in F. Then F is the centre, and F P the radius, of the required circle.

(2.) When the given circle is to be included, the length of the radius of the given circle must be measured off below the given point as P D′. Join C D′, and proceed as before. F′ is the centre of the required circle.

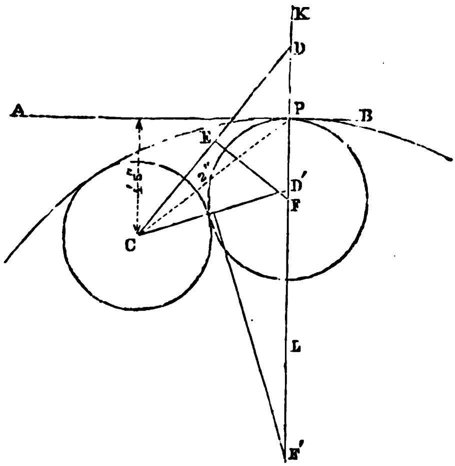


Fig. 78.

Problem LXXXIV.

The line and circle being given as in the preceding problem, required a second circle to be tangent to the line and to touch the former at a given point P in its circumference.

(1.) Join the centre C to the given point P, and produce it beyond P. Draw D P perpendicular to C P, meeting A B in D. With D as centre, radius D P, describe

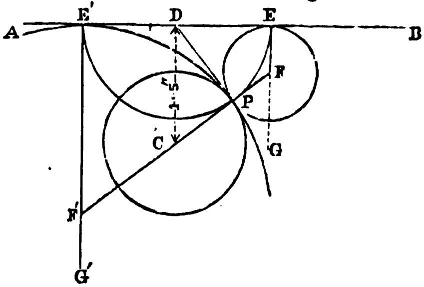


Fig.

the arc D P E, meeting A B in E. Draw E G perpendicular to A B. The point F, where the lines E G and P C produced intersect, is the centre of the required circle.

(2.) When the given circle is to be included, the construction must be modified, in that the arc D E′ must be described in the opposite direction, and P C must be produced beyond C to intersect the perpendicular from E′. Then F′ is the centre of the required circle.

Problem LXXXV.

Two Points, A *and* B, *are* 2″ *apart.* A *is* 1·5″ *and* B ·8″ *from a straight line,* C D. *A Circle is required which shall pass through the points* A *and* B, *and touch the given line* C D.

To determine the position of the points A and B, draw two lines parallel to C D at the respective distances of A and B from the given line.

Mark A upon the line which is 1·5″ from C D, and, with A as centre, radius 2″, intersect the other parallel in B.

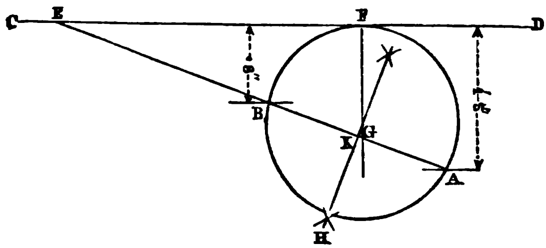


Fig. 80.

Join A B, and produce it beyond B to meet C D in E. A mean proportional between A E and B E will determine C′ F′. Set the length of E′ F′ along the line C D from E to F. This latter point is that at which the required circle will touch the given line.

Bisect A B in K, and draw G H perpendicular to A B, intersecting a perpendicular to C D from F in the point G, which is the centre of the required circle.

PROBLEM LXXXVI.

A Point, P, *is* 2″ *from the centre of a Circle of* ·75″ *radius. Required a line drawn through* P, *and cutting the circle in* Q *and* R, *so that the intercepted segment,* Q R, *of the line shall be* 1″ *in length.*

Take a length equal to Q R (1″) in the compass, and mark off a chord, A B, equal to it. Join A B. From C draw C D perpendicular to A B. With C as centre, radius C D, describe a circle. Then from P draw P Q R tangential to this circle, and it will cut the given one in two points, Q and R, so that Q R will be 1″ long.

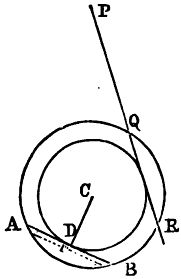


Fig. 81.

This is evident, as all chords in the same circle and equidistant from the centre are equal to one another (*Euclid*, Bk. III., Def. 14).

PROBLEM LXXXVII.

In a Quadrant (quarter of a circle) to inscribe a circle.

Make two lines, A B and A C, perpendicular to each other, and with A as centre, radius A B, describe the arc B C. Then A B C is a quadrant.

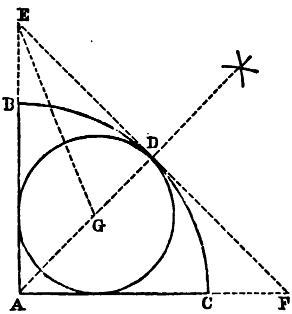


Fig. 82.

Bisect the angle, B A C, by the line A D. Draw E F through D, perpen-

dicular to A D, and meeting A B and A C, produced in E and F. Bisect the angle, A E D, by the line E G, meeting A D in G. Then G is the centre of the required circle.

PROBLEM LXXXVIII.

Two Points are 1·75" and 2·25" from the centre of a Circle of 1" radius, and 2" from each other. Draw the circle which, passing through these two points, shall touch the given circle.

To place the given points, &c., in position, draw the circle, C, of 1" radius, and with the same centre draw two arcs, radii 1·75" and 2·25" respectively. Take 2" in the compass, and, setting one leg upon any point in one of these arcs, draw a circle intersecting the other. These two points, A and B, will then be those given in the question.

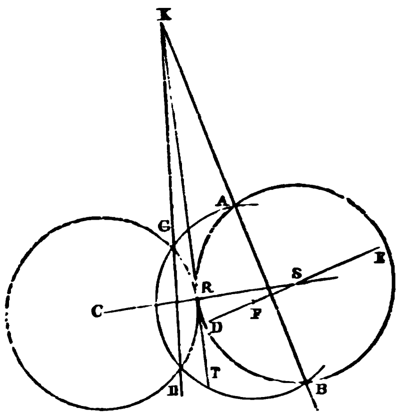


Fig. 83.

Join them and draw D E through the middle point of A B, and perpendicular to it. With any point, F in E D, as centre, describe an arc, passing through A and B, and intersecting the given circle in G and H.

Join G H, and produce it beyond G, to meet A B produced beyond B in the point K.

From K draw K T tangent to the given circle. Then a line from C perpendicular to the tangent, K T, will,

if produced, meet E D in S, which will be the centre of the required circle, and A S or B S will be the radius.

EXERCISES.

1. The centres of two circles are 2·5″ apart; their radii are ·75″ and ·5″. Draw the four lines touching both circles.

2. Draw two lines crossing each other at an angle of 60°. Describe circles of ·5″ and ·75″ radii in the opposite acute angles, and circles of 1″ and 1·25″ radii in the opposite obtuse angles, all these circles to touch both lines.

3. A line is one inch from the circumference of a circle of 1″ radius; draw a circle to touch the given one and the line, from any two of the following conditions:—

   a. The circle to be 1″ radius.

   b. To touch the line in a point 2·75″ from the centre of the given circle.

   c. To touch the given circle in a point 1·75″ from the line.

4. Draw three circles of 1, 1·25, 1·75 inches radii, each circle touching the other two.

5. The distance between the centres (A) (B) of two circles is 2″; their radii are ·75″ and 1″ inch; draw a circle of 2″ radius to touch both the former, but to contain the smaller within it; the points of contact to be correctly determined

6. Draw two lines at an angle of 40°. Draw two circles, each touching the lines and one another, the radius of the smaller one to be 1″.

7. Two points, A and B, are 2″ and 3″ from the centre of a circle of 1″ radius, and 2·75″ from each other. Describe a circle to touch the given one, and to pass through A and B.

CHAPTER VIII.

ON SCALES.

It is very seldom that drawings can be made equal in size to the objects themselves.

Sometimes, as in representations of parts of buildings, the drawings must be considerably smaller than the parts they depict. At other times, as in representations of the mechanism of a watch, they must be much larger to show with sufficient accuracy the minute details.

In both cases it is necessary that all parts of the drawing should bear the same relation to the size of the corresponding parts in the object.

To effect this, a scale is constructed which consists of a line, very accurately divided, in such a manner as to represent in a smaller space, the standards of length used in measuring the original object.

Thus a line 6″ long can be divided into 36 equal parts, and each part can be assumed to represent one inch. Then the whole line will represent one yard. Again, 12 of these parts will indicate 1 foot.

In this case, it is clear, the original yard is six times its length as shown upon the scale; or expressing the same fact in another form, the scale is $\frac{1}{6}$ of the original.

This fraction ($\frac{1}{6}$) is called the *representative fraction* of the scale.

If a line 2″ long be intended to show 1′ 6″, the representative fraction is $\frac{1}{9}$.

Scales are of no service, except they be very accurate. It is necessary, therefore, that the student should construct them with great care, and also that he should thoroughly test them before he uses them in making a drawing.

The same lengths should be taken upon different

parts of the scale, so as to prove that the divisions are uniform.

A scale of merely equal parts is the simplest form. This is called a *plain scale.*

Diagonal scales are used for very minute divisions.

The following measures of length should be committed to memory:—

12 inches make 1 foot.
3 feet, or 36 inches make 1 yard.
$5\frac{1}{2}$ yards make 1 pole.
4 poles, or 22 yards make 1 chain.
10 chains, or 40 poles, or 220 yards make 1 furlong.
8 furlongs make 1 mile.
3 miles make 1 league.

Problem LXXXIX.

To construct a "Plain Scale" of $\frac{1}{60}$ to represent yards and feet. The scale to be long enough to measure 5 yds.

As the scale is $\frac{1}{60}$ part of the original, and it is to be long enough to show 5 yards, the whole of it will really be $\frac{5}{60}$ yard long. This will be $\frac{180''}{60}$, which is 3″. Draw a line therefore, 3″ long, and divide it into 5 equal parts. Each of these will then represent 1 yard. The first division on the left must be then divided into 3 equal parts to show feet. It is usual to number the divisions as shown in the figure. That is, having assumed that the division to the left, shows feet, the yards are numbered towards the right.

As an instance of the method of using such a scale, we will suppose that 13 feet is to be measured. As this is

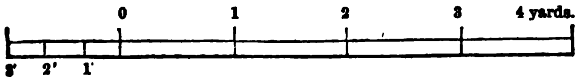


Fig. 84.

4 yards 1 foot, we must place one leg of the dividers at the
1 E. F

point 4, and the other at 1′. The distance between these two figures will represent 13 feet.

Problem XC.

Draw a Scale of 12·5 yds. to 1″. It must be long enough to measure 35 yds.

As 1″ is to represent 12½ yards, 2″ will represent 25 yards. Take therefore a line 2″ long, and divide it into 5 equal parts. Each of these divisions will represent 5 yards. The first two must again be divided each into 5 equal parts. The result will be that 10 smaller divisions will be obtained, each showing on the scale 1 yard. The whole line will require to be made longer to

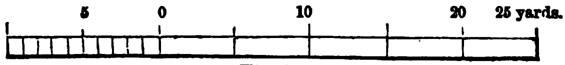


Fig. 85.

represent 35 yards. Add therefore to it a length equal to that which shows 10 yards, as it already represents 25.

Problem XCI.

Draw a "Plain Scale," in which 3″ shall represent a real length of 1 chain, (22 yards.)

The first part of this problem is similar to the preceding. When the line 3″ long has been divided into 22 equal parts, each part will represent 1 yard. The

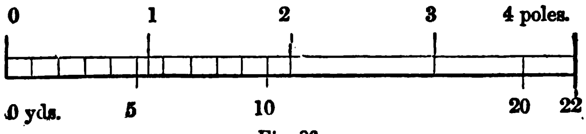


Fig. 86.

whole line must then again be divided into 4 equal parts, to show poles. The divisions will of course, if correct, be equal to $5\frac{1}{2}$ yards.

PROBLEM XCII.

To show, by diagonal division, Tenths and Hundredths of 1″.

Divide a line 1 inch long into 10 parts, in points 1, 2, 3, &c. At each extremity of the line erect a perpendicular, and mark off upon one of these 10 equal distances, numbering them as in the diagram. Through each of these latter divisions draw horizontal lines, and join C to 9. Then parallels to C 9, through each of the divisions of the 1′-line, will complete the necessary construction.

To measure off ·47 of 1″, place one leg of the dividers upon the point E, which is on the horizontal line 7, and the other leg at the intersection of the diagonal line 4 with it. The student will see that the distance he measures is rather more than four-tenths, as shown upon the line A B, and less than five-tenths. The horizontal line 7, by its intersection with the diagonal through 4, gives the exact difference for the seven-hundredths.

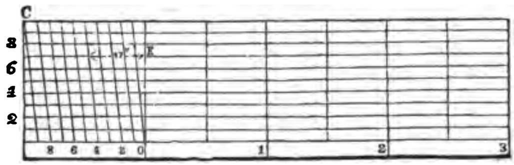


Fig. 87.

EXERCISES.

1. Construct the following scales, to show yards and feet:—

$\frac{1}{16}$ to be long enough to measure 10 yards.
$\frac{1}{50}$ " " " 30 yards.

2. Construct the following scales, to show feet and inches:—

$\frac{1}{12}$ to be long enough to measure 8 feet.

$\frac{1}{18}$ " " " 10 feet.

3. A line 17·5 yards long is represented on a certain drawing by 3·5″. Construct the scale to show yards.

4. If 2 chains be represented by 4″, construct the same scale, to show poles and yards.

5. Determine $\frac{1}{55}$ of an inch by diagonal division.

SOLID GEOMETRY.

CHAPTER I.

INTRODUCTION, POINTS, LINES, ETC.

(This Chapter is to be studied simultaneously with Chapter II.)

SOLID or Descriptive Geometry enables us to represent the three dimensions of solids—length, breadth, and thickness—in two drawings. Thus, if the shape of the base of an ordinary rectangular instrument box be drawn upon the paper, it will indicate the length and breadth of the object, but not the height. To show the height, a drawing must be made of the front or end of the box.

Its base when in the ordinary position upon the table is horizontal, and the front is vertical; or, in other words, the base is part of a horizontal plane,* and the front, part of a vertical plane.

Thus we see that by representing the object as it appears, first upon a horizontal plane, and afterwards upon a vertical one, we can show its three dimensions.

* A plane is a perfectly level surface, like that of standing water.

Euclid defines a plane as "that upon which, any two points being taken, the straight line which joins those points lies wholly in that plane."

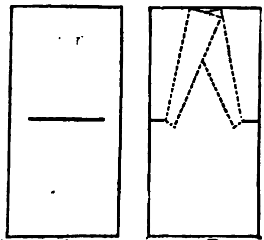
The student will understand that the views above described are those which would be seen by an observer placed at an infinite distance from the object. The rays of light by which the box is perceived, are supposed to be parallel, and not to converge towards the eye, as they actually do.

The drawing upon the H. P. (horizontal plane) is called the PLAN; that upon the V. P. (vertical plane), the ELEVATION.

Many elevations of one object can be made by assuming as many different positions for the vertical planes. By this means, end or profile views, front elevations, &c., can be obtained.

If we wish to show the arrangement of the internal parts of our box, we must suppose the object, cut by another plane in such a manner as to expose the required parts, and a drawing must be made of those parts upon that plane. Such a drawing is called a SECTION.

The horizontal and vertical planes are called co-ordinate planes, and their intersection is a straight line, which receives the name of ground-line, intersecting line, &c.



A B

Fig. 88.

In this work, this line will be distinguished by the letters X Y.

A model of these planes can be made in a very simple manner:

Cut two pieces of card-board in the way indicated by the dark lines in figure 88, A and B. Then fold one of these pieces upon the lines shown, as dotted in figure 88B, and pass the folded piece of card-board half-way through the slit in the other piece. Unfold it, and a model of the co-ordinate planes with their intersection will be the result (fig. 89).

It will be seen that the two pieces of card-board will make four angles or corners. Such angles as these are called dihedral angles, to distinguish them from rectilineal angles.

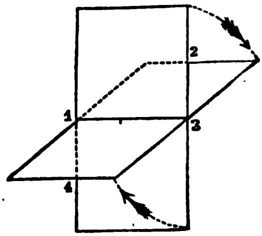


Fig. 89.

If the model be so held as to present to the holder a full view of the v. p., and the end of a pencil be imagined to represent a point, it will be noticed that there are several distinct positions which it can assume with regard to the co-ordinate planes. A point may be in *front* or *behind* the v. p., *above* or *below* the h. p., or in either or both of them.

The four dihedral angles are named as 1st, 2nd, 3rd, and 4th, according to the following table:—

A point in front of the v. p. and above the h. p. is in the first dihedral angle.

A point behind the v. p. and above the h. p. is in the second dihedral angle.

A point behind the v. p. and below the h. p. is in the third dihedral angle.

A point in front of the v. p. and below the h. p. is in the fourth dihedral angle.

If two lines be supposed to pass through the actual position of a point, one being perpendicular to the h. p., and the other to the v. p., the intersection of these lines with the respective planes will be the plan and elevation of the point. These imaginary lines are called *projectors*, and the plan and elevation, the projections of the point.

Thus a point in the first dihedral angle would have its plan upon the front part of the h. p., and its elevation upon the upper part of the v. p. A point in the third angle would have its plan upon the back part of the h. p., and its elevation upon the lower part of the v. p.

The student, to understand these facts, must use his model, and demonstrate for himself the positions of the various plans and elevations.

If a point be *in* * one of the co-ordinate planes, either its plan or elevation must be in X Y; and if it be *in both* planes, the plan and elevation will coincide with the actual point itself upon X Y.

In making drawings upon our paper, only one surface is used to represent both the co-ordinate planes; and for this purpose the v. p. is supposed to rotate upon X Y as a hinge, whilst the h. p. remains stationary. The model will best illustrate this. The piece of card-board which represents the v. p. should be rotated in the direction shown by the arrows in figure 89, until the upper part of the v. p. coincides with the back part of the h. p., and the lower part of the v. p. with the front part of the h. p.

All elevations which occur above X Y will be above X Y after this rotation.

When the co-ordinate planes are thus made to coincide, the representations of the projectors, which determine the plan and elevation of a point, will be found to form one straight line perpendicular to the ground line. Hence the rule that *plan and elevation of a point always fall in the same straight line, perpendicular to* X Y.

The following nomenclature is adopted to recognize points and their projections.

A point itself is indicated by a capital letter, as A, B, &c.

The plan is denoted by an italic letter, as *a*, *b*, &c.

The elevation by an italic letter with a dash, as *a'*, *b'*, &c.

Problem I.

To determine the projections of the points A B C D E *and* F, *when in the following positions :—*

* When a point or line forms a part of a plane, that plane is said to contain the point or line.

A. 1·8" *in front of the v. p. and* 1·6" *above the h. p.*
B. 1·4" „ „ 1·8" *below* „
C. 1" *behind* „ 1·7" *below* „
D. 1·5" „ „ ·5" *above* „
E. 1" „ „ *and in the h. p.*
F. *In both planes.*

Draw an indefinite projector, because the plan and elevation of point A must be upon the same line, perpendicular to X Y. The point being 1·8" in front of the v. p., and 1·6" above the h. p., the plan will fall upon the front portion of the h. p., 1·8" from X Y, and the elevation upon the upper portion of the v. p., 1·6" from X Y.

When the co-ordinate planes are made to coincide, the plan and elevation will be below and above X Y respectively.

Mark these distances upon the projector, and letter

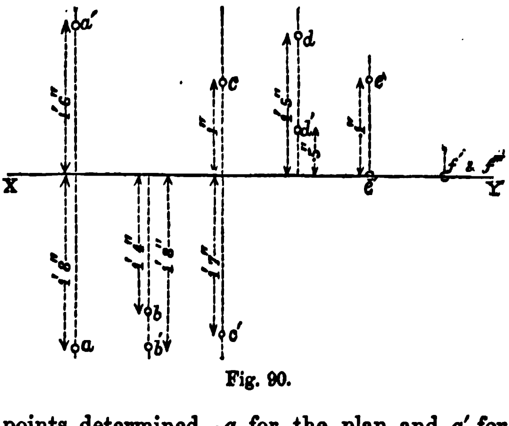


Fig. 90.

the points determined—*a* for the plan and *a'* for the elevation. This should be reasoned out by means of the card-board model.

The determination of the projections of points B C and D will present no particular difficulties, if the above be understood.

Point E is to be *on* the h. p., consequently its elevation will be *in* the ground-line. The plan is determined by its position in relation to the v. p., as before.

Point F being in *both* planes, its actual position must be upon the ground-line, and its projections must coincide with that position.

Problem II.

Given the projections of points A, B, *and* C, *to determine their position in relation to the Co-ordinate Planes.*

This problem is the converse of the preceding one.

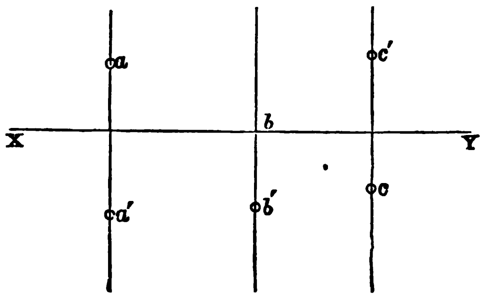


Fig. 91.

The model should be used to illustrate the method of discovering the actual position of point A. Its plan being above X Y, the point itself must be behind the v. p., and the distance of the plan from X Y is the actual distance of the point behind the v. p. Similarly, its elevation being below X Y, the point must be below the h. p., and the length of the projector which determines the elevation is the real distance of the point from the h. p.

Point B is below the h. p., but in the v. p., as the plan falls upon X Y.

Point C is above the h. p., and in front of the v. p.

PROBLEM III.

Having given the projections of a point A, to determine its distance from X Y.

By distance from X Y is meant the length of the

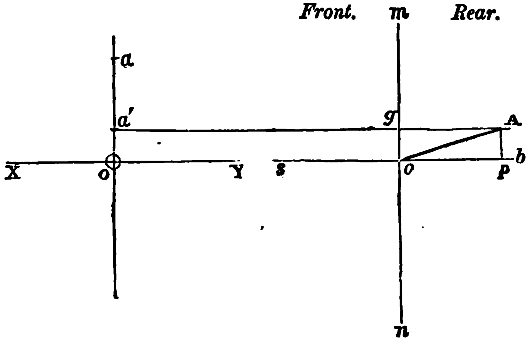


Fig. 92.

diagonal of a rectangle, made up of the two projectors of the point, and their representations upon the co-ordinate planes.

To determine this, a profile view must be taken of those planes. The projections of the point will show that its actual position is above the h. p. and behind the v. p.

Draw two lines, *s t* and *m n*, perpendicular to each other, to represent a profile view of the co-ordinate planes, and assume one side to be the front and the other the rear of the v. p., as in fig. 92. Measure the distance *o a* (fig. 92) along *o t* (fig. 92), to *p*, and the distance *o a′* along *o m*, to *g*. Draw *g* A and *p* A perpendicular to *o m* and *o t* respectively, and join *o* A. Then *p* A *g o* represents the rectangle mentioned above, and A *o* is its diagonal, or the real distance of the point A from X Y.

Problem IV.

A point P *is* 1" *in front of the v. p. and below the h. p., its distance from* X Y *is* 1.5"*; determine its plan and elevation.*

This is the converse of the preceding problem. A profile view must be made of the co-ordinate planes, and an indefinite line must be drawn parallel to the line which represents the v. p. 1″ in front of it. The point *o*, which indicates X Y, must be used as centre for an arc of 1·5″ radius, cutting the indefinite line in a point below that which represents the h. p. The distance of this intersection below that line is the real distance of the point *p* below the h. p.

Now, having the real position of the point as regards both planes, proceed to find its projections as before.

We have found that points may occupy positions in either of the four dihedral angles, and that their projections will determine those positions.

Lines also can occupy similar positions with regard to the co-ordinate planes, and their projections will follow the same laws as those of points.

But lines present another consideration to our minds. They may be horizontal, perpendicular, or oblique.

If a pencil be held in such a position that it is parallel to the h. p., its projection upon that plane will be equal in length to the pencil itself. Similarly, if the pencil be held parallel to the v. p., its projection upon that plane will be of exactly the same length.

But if the pencil be held in any other position, either its plan or its elevation, perhaps each of them will be represented by a shorter line.

When the pencil is held parallel to the v. p., but not parallel to the h. p., it is said to be *inclined* to the latter. Its elevation is, under these circumstances, equal in

length to the line itself; but its plan is shorter, and the greater the inclination the shorter the plan, until at length, when the pencil stands vertically, its plan becomes a point.

If the elevation of a line be shorter than the line itself, whilst the plan remains of the same length, the line is inclined to the v. p., but parallel to the h. p.

When a line is inclined to both planes each of its projections is shorter than the line itself.

The inclination which a line makes with either of the co-ordinate planes is measured by the angle which the line makes with its projection upon that plane. Thus, the angle which a line makes with its plan is its inclination to the h. p., and the angle which it makes with its elevation, its inclination to the v. p.

It is usual to indicate the inclination of a line to the h. p. by the Greek letter θ (theta), and that to the v. p. by ϕ (phi), when the actual number of degrees is not required.

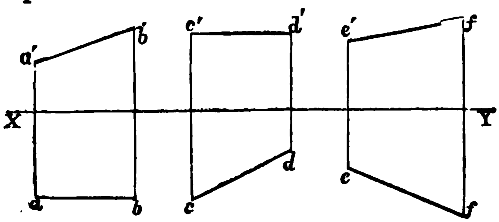


Fig. 93.

In fig. 93 the projections of three lines are shown.

The line A B is inclined to the h. p., but is parallel to the v. p.

The line C D is inclined to the v. p., but is parallel to the h. p.

The line E F is inclined to both planes.

Problem V.

To determine the projections of a Line, A B, 3″ *long, which is parallel to the v. p., and* 1″ *in front of it; its extremities being* ·5″ *and* 1·2″ *above the h. p. respectively.*

As the line is parallel to the v. p., its full length and inclination will be shown in the elevation. Draw two

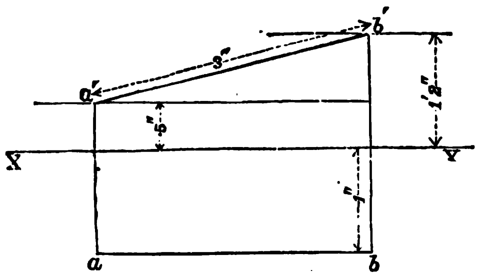


Fig. 94.

parallel lines ·5″ and 1·2″ above X Y, and assume any point upon one of them as the elevation of the extremity, A. With *a′* as centre, and radius equal to the length of the line, describe an arc, intersecting the remaining parallel in *b′*, which will be the elevation of the other extremity. Join *a′ b′* to complete the elevation.

The plan will be a line parallel to X Y, and 1″ below it; the points *a* and *b* being determined by projectors from *a′* and *b′*.

Problem VI.

A Line, A B, 3″ *long, is inclined to the h. p. at* 36°, *and its plan makes an angle of* 20° *with* X Y. *Show the elevation.*

In such problems as these, it is convenient to imagine

the line as lying upon the surface of a cone.* If it have one extremity in the apex of that cone, the surface of the solid will be the *locus* of all lines having the same inclination as the side of the cone. Commence this problem by drawing a line, B a', upon the v. p., at an angle of 30°, with X Y, and 3″ long. A projector from a' will determine upon X Y the point a, which would be the centre of the base of a cone upon which the line must lie. Describe the arc B b c, as in the figure. This arc will

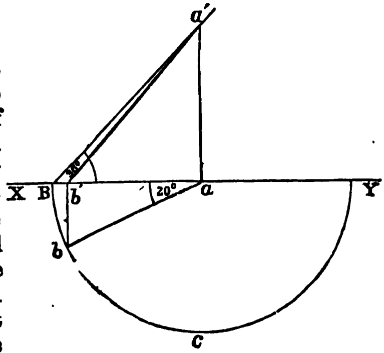


Fig. 95.

represent part of the plan of the base of the solid; and the radii of this curve would be the plans of all the lines which lie upon the surface of the cone, having their extremities in the apex and the base of the solid. Draw $a\,b$ at an angle of 20° with X Y, and it will be the plan required. A projector from b will determine the point, b', and a' b' will be the required elevation.

Problem VII.

Given, the projections of a Line, A B, *to determine* (1) *its traces;* (2) *its actual length; and* (3) *its inclinations to both planes of projection.*

If a line makes an angle with a plane, it either penetrates that plane, or would do so if it were produced far

* For description of cone, see Chapter II. (Solid Geometry.)

enough. The point where the penetration takes place is called the *trace* of the line upon that plane. The horizontal trace (*h. t.*) of a line, therefore, is the intersection of that line with the h. p.; and the vertical trace (*v. t.*) of a line is its intersection with the v. p.

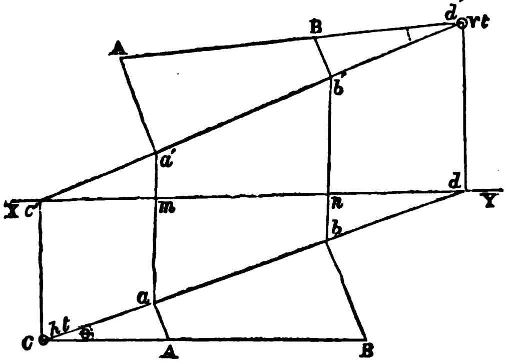


Fig. 96.

If a line be parallel to either plane, it will of course have no trace upon it.

Let *a′ b′*, *a b* be the projections of a line, A B. It is inclined to both the co-ordinate planes, and will therefore have a trace upon each of them, if produced. To discover the position of the h. t., produce the elevation until it intersects X Y in the point *c′*. Then the line *b′ c′* is the elevation of the line which penetrates the h. p. in the point C. If a projector be drawn from *c′* until it meets the plan, *a b*, produced in *c*, that point will be the h. t. required.

The v. t. of the line is determined by a similar construction. The plan must be produced beyond *b*, until it meets X Y in *d*; and a projector from that point, meeting the elevation produced in *d′*, will give the required v. t.

As both the projections of the line make angles with

X Y, neither of them is as long as the line itself. To determine the true length of A B, the following construction is necessary:—Conceive the real line as supported in its position above the h. p. by its vertical projectors, and the whole arrangement to revolve upon the plan as an axis, until it coincides with the h. p. This is called "*constructing*" the line into the h. p. The points A and B will then be upon the lines, A *a*, B *b*, perpendicular to the plan, *a b*, at distances equal to the heights of those points, as shown in the elevation. Thus the length *a* A is equal to *m a′*, and *b* B to *n b′*.

Then A B is the required real length of the line.

If A B be produced beyond A, it will pass through the h. t., and the angle which it makes with the plan will be the inclination of the line to the h. p.

In the figure the line is shown also "constructed" into the v. p. by similar means. It is produced beyond B′, and passes through the v. t. of the line, the angle shown being the inclination of A B to the v. p.

Note.—When the inclinations are small, it sometimes happens that the traces fall without the paper; but in that case the true length can be determined as above, and a parallel to the projection used, intersecting the true length found, will give the angle of inclination.

It is important that this problem should be well understood, as it is frequently necessary to determine the inclinations of edges of solids in the midst of complex drawings. The construction explained above offers a very ready means of doing this, although other methods are sometimes preferable.

PROBLEM VIII.

A Triangle, A B C, *is represented in plan by an equilateral triangle of* 1″ *side. The points* A, B, *and* C *are* ·5″, 1·2″, *and* ·8″ *above the paper, respectively. What is the true shape of the figure, and the inclination of the side,* A B?

The real length of the sides of the triangle can be de-

termined by the construction explained in the preceding problem from their projections. The triangle made up of the lengths thus found will be the true shape of the figure. The elevation need not necessarily be drawn, as the heights of the points are given in the question. Set

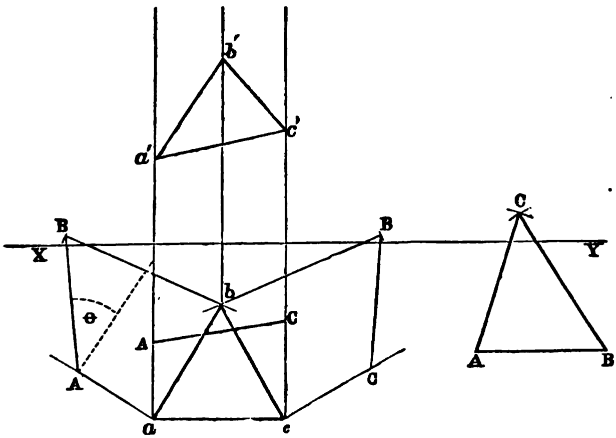


Fig. 97.

out perpendiculars from the extremities of each of the three lines in plan. Mark off along these perpendiculars distances equal to the heights of the points. Join as shown in diagram. Then A B, B C, C A are the real lengths of the sides of the triangle. The figure, A B C, is the one required, whose sides are equal to these three lines. The inclination of the line A B is shown by the angle θ which A B makes with its plan.

Problem IX.

To determine the plan and elevation of a Line (length at pleasure) inclined at 60° to the h. p., and 20° to the v. p.

If a number of lines lie upon the surface of a cone which stands with its base upon the paper, each having

one of its extremities in the apex, these lines will all be equally inclined to the horizontal, and moreover make with the paper the same angle which the sides of the cone make with its base.

In fact, the surface of a cone whose base angle is 60° is the locus of all straight lines which pass through the apex, and are inclined at that angle to the h. p.

At any point A, therefore, in X Y, draw a line, A *b′*, making an angle of 60° with it. Draw *b′ b* perpendicular, and consider A *b′ b* as half elevation of a cone. Then an arc, having *b* for its centre, and *b* A for radius, will represent part of its plan.

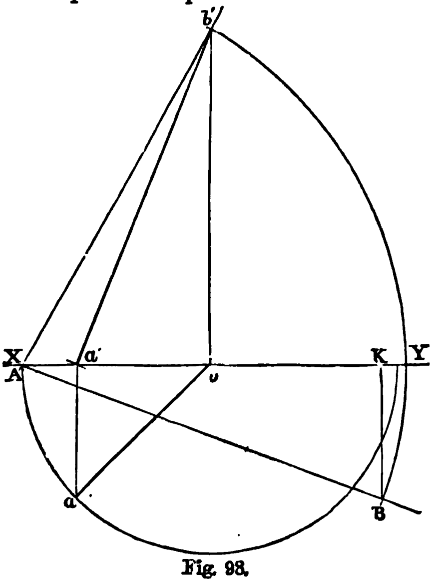


Fig. 98.

Now, if lines through B be conceived to lie upon the surface of the solid, only two of them—viz., those on the extreme right and left—will be shown in their full length in elevation. As the line travels round the

solid, its elevation alters in length. When it is in such a position as this, it makes an angle with the v. p.

Our problem, therefore, is to determine the exact position upon the cone, when the line is inclined 20° to the v. p. There are, of course, four solutions—two when the line is in front of the cone, and two when behind.

At the point A, set out a line A B, as long as the side of the cone, and making an angle of 20° with X Y. Draw B K perpendicular to the ground line, and the length, A K, is that of the elevation of the line when it makes an angle of 20° with the v. p. With *b'* as centre, radius equal to A K, describe an arc intersecting X Y in *a'*. Join *a' b'*. This is the required elevation. A projector from *a'* meeting the arc first drawn in *a* gives the plan of one extremity, A, of the line. Join *a b*, and the problem will be solved.

This is only one of the solutions; but the other three are not very difficult to determine, if the above be understood.

PROBLEM X.

A line A B, 3″ *long, has its extremity,* B, *in the h. p., and* ·75″ *in front of* X Y; *its extremity,* A, *is in the v. p. The inclination of the line is* 40°. *Draw plan and elevation.*

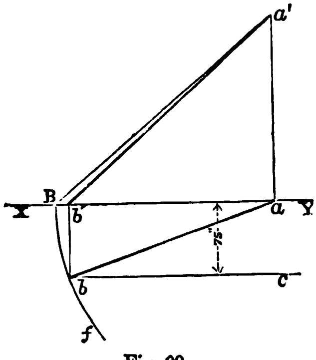


Fig. 99.

At any point, B, in X Y, make the line B *a'* at an angle of 40°. B *a'* must be 3″ long. Draw *a' a* perpendicular to X Y, and describe the arc B *f*, with *a* as centre. Make *b c* parallel to X Y, at a distance ·75″ from it, intersecting the arc B *f* in *b*. Then *b a* is the plan of the required line, and its elevation, *b' a'*, is determined by

drawing a projector through b to meet X Y in b', and joining b' a'. In this construction a cone has been assumed, having a base angle of 40°, the apex a' of the solid being in the v. p., and its sloping side 3″ long. The line in the question was then supposed to lie upon the surface of this cone, having one extremity in the apex and the other in the edge of the base, at a point ·75″ from the ground line. By this means it was ensured that the ends of the line should be in the co-ordinate planes, as required.

EXERCISES.

1. Show the plans and elevations of the following points, using the same ground-line for all. The distance between the lines joining the projections in each case may be 1″ :—

A, 3″ above	the horizontal	plane,	1·8″ before	the vertical	plane.
B, 2″ above	,,	,,	3·5″ behind	,,	,,
C, 2·5″ below	,,	,,	·4″ before	,,	,,
D, in	,,	,,	2·5″ before	,,	,,

2. a b, two inches apart, are the *plans* of two points, of which A is 1·7, B 3· inches above the paper. What is the length, and inclination to the paper, of the line A B?

3. Draw the *plan* of a line three inches long when inclined at 40° and an elevation of it on any vertical plane not parallel to the line.

4. A line 3·5″ long is to be represented by a plan and elevation according to the following conditions :—

   a. When inclined to the paper at 60°.

   b. When its ends are 1″ and 2·5″ above the paper.

5. The plan of a line is 2″ long and its elevation is 3″. The *projectors* of its extremities are 1″ apart, measured along X Y. What is its true length and inclination?

6. Draw the plan and elevation of a line (length at pleasure) which is inclined at 30° to the horizontal, at 40° to the vertical plane.

7. Draw the plan and elevation of a point A, which is situated above the horizontal plane, 2″ behind the vertical plane, and is 3′ distant from X Y.

8. A square of 1″ side is the plan of a certain quadrilateral figure. The angular points are ·7″, 1·2″, 1″, and ·9″ above the horizontal plane. What is its true shape?

9. A line A B, 3″ long, has its extremity, A, in the vertical plane, at a height of 1·8″. Its other extremity, B, is ·5″ above the horizontal plane. Draw its projections.

10. A line, A B, 3″ long, is inclined 50° to the horizontal plane. Draw its projections when its plan makes an angle of 30° with X Y.

CHAPTER II.

ELEMENTARY PROBLEMS ON SOLIDS.

N.B.—(*The problems of this chapter are to be studied simultaneously with those of Chapter I.*)

THE following are the solids commonly used to illustrate the principles of Solid Geometry:—

The **cube, prism,** and **pyramid**; the **sphere, cone,** and **cylinder.**

A **cube** is a solid having six equal faces, all squares.

A **right prism** is a solid having two equal and similar bases, with rectangular faces perpendicular to them.

If the faces be not perpendicular to the bases, the prism is oblique.

A **right pyramid** has one base and a number of triangular faces, meeting in a point over the centre of that base. This point is called the apex.

If the apex be not over the centre of the base, the pyramid is oblique.

Prisms and pyramids are named from the shapes of their bases, as square prism, hexagonal pyramid, &c.

A **sphere** is a solid whose surface is at all parts equidistant from a point within it, called the centre. If a semicircle revolve upon its diameter, it generates the surface of a sphere. All plane sections of a sphere are circles.

A **cone** may be defined as a pyramid with an infinite number of faces. It has a circular base, and its surface is generated by the revolution of a right-angled triangle upon its perpendicular, as an axis.

A **cylinder** bears a similar relation to a prism that a cone does to a pyramid. It is a prism with an infinite number of faces. Its bases are circles, and its

surface is generated by the revolution of a rectangle upon one of its sides.

The **axis** of a prism or cylinder is a line joining the centres of the bases.

The **axis** of a cone or pyramid is a line joining the centre of its base to the apex.

A **tetrahedron** is a solid having four faces, each being an equilateral triangle.

An **octahedron** has eight faces, all equilateral triangles. It consists of two square pyramids, placed base to base (the height of each pyramid being equal to half the diagonal of the square).

Models of these solids should be seen by the student; and no teacher should be without them. Those made of wire, painted white, are the best, as they are so readily seen at a distance, when placed before a black board.

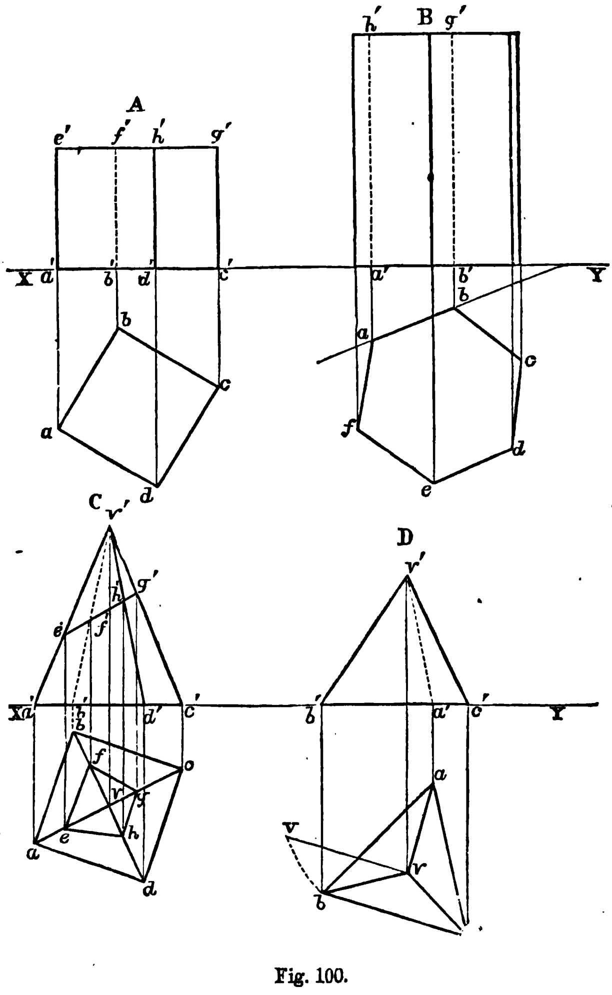
In figures 100 and 101 plans and elevations of these solids are shown, which it is most advisable the student should work out for himself.

Fig. A is the plan and elevation of a cube when it stands with its base upon the h. p. Its plan, *a b c d*, is a square which should be drawn first. In this figure it is assumed that none of the faces of the solid are to be parallel to the v. p. Each corner of the square is the plan of one of the perpendicular edges of the cube, and *a b c d* is that of the upper surface also.

Projectors from *a b c d* will determine upon X Y the elevations of the four corners of the lower square. The perpendiculars, $a' e'$, $b' f'$, $c' g'$, and $d' h'$ must next be drawn, equal in length to the edge of the cube, and a line passing through $e' f' g' h'$ will complete the elevation.

The line $b' f'$ must be a dotted one, as it represents the perpendicular edge, which would be hidden from an observer placed in front of the object.

Fig. B is the plan and elevation of a hexagonal prism standing with its base upon the h. p. The construction of the projections is similar to that of the cube. It should be noticed that the angle which the line *a b*


A
e′
f′
h′
g′
X
a′
b′
d′
c′
b
c
a
d
B
h′
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a′
b′
b
Y
a
c
f
d
e
C
v′
g′
h′
e′
f′
X
a′
h′
d′
c′
b
f
c
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D
v′
b′
a′
c′
Y
a
V
v
b

Fig. 100.

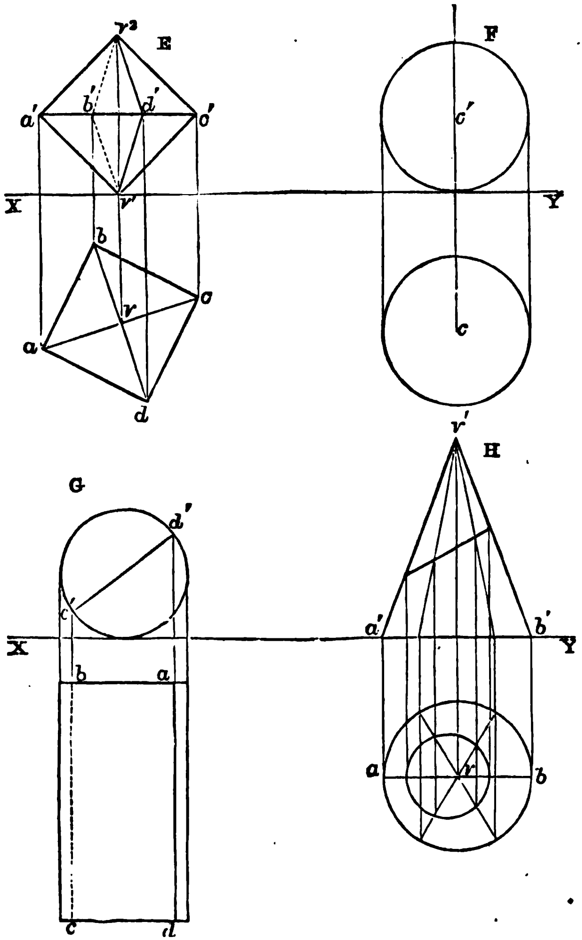


Fig. 101.

makes with X Y is the angle of inclination of the face, A B G H, to the v. p.

Fig. C is the plan and elevation of a pyramid with its base upon the h. p. Commence by drawing the square *a b c d*, and join the points *a c* and *b d*. This completes the plan, as the square is the plan of the base of the solid, and the diagonals that of the sloping edges. The point *v*, where the diagonals intersect, is the plan of the apex of the pyramid. Projectors from the four corners of the square will determine the elevation of the base upon X Y. That through *v* will contain the elevation of the vertex, and its exact position is obtained by measuring the height of the pyramid above X Y. The line *v' b'* must be dotted.

A section line *e' g'* is shown in the figure. This line is the elevation of a section made by a cutting plane. It is clear that the points *e' f' g' h'* in that line are the elevations of the points where this cutting plane intersects the sloping edges of the pyramids, and as plan and elevation are always in the same straight line perpendicular to X Y, the plans of these points can be found by drawing projectors through *e' f' g' h'* until they intersect the plans of the sloping edges in *e f g* and *h*. Join these and the plan of the section will be determined. This is not the real shape of the section. A method for determining this will be given in the next chapter

Fig. D is the plan and elevation of a tetrahedron resting with one of its faces upon the h. p. Its plan is an equilateral triangle, having lines joining each of its angular points to the centre of the figure. The point *v* is the plan of the axis of the solid. The elevation of the base is determined as before, and a projector *v'v* will contain the elevation of the apex; but as the height of the figure is not given, depending as it does upon the length of the edge, a special construction is necessary for determining it.

The plan of a sloping edge, that edge itself and the axis of the solid together form a right-angled triangle. The plan is the base, the axis is the perpendicular, and the sloping edge, the hypotenuse.

Now, if this triangle be imagined to revolve upon its base, until it is horizontal, it is clear that the height of the figure will be shown in the perpendicular of the triangle.

Set out, therefore, from *v*, a line *v* V, perpendicular to *a v*, and with *a* as centre, radius *a b*, describe an arc *b* V, intersecting the perpendicular in V. Then *v* V is the height of the tetrahedron. The completion of the elevation presents no further difficulty.

Fig. E is the plan and elevation of an octahedron having its axis vertical. The plan of it is the square *a b c d*, with its diagonals. The octahedron, as can be seen by a model, has three equal diagonals or lines upon which the solid can revolve. Two of these diagonals are shown in the plan, and the third is the vertical axis. As these are all equal, it only remains for us to measure either of the diagonals *a c* or *b d*, and use the length found as the height of the elevation. The elevation of the square will be a straight line a' b' c' d' mid-way between the apices v^1 and v^2.

Fig. F is the plan and elevation of a sphere. Both are circles; further explanation is unnecessary.

Fig. G is the plan and elevation of a cylinder when it rests with its side upon the h. p. and with its base, parallel to the v. p. In this case it is necessary to commence with the elevation which is a circle tangent to the ground line. The plan is a rectangle. A section is shown in elevation by the line c' d' and its plan, *a b c d* is determined from the elevation by projectors as before.

Fig. H is the plan and elevation of a cone, resting with its base upon the h. p. A section of this solid is given and the plan shown, which is determined by supposing the figure to be a pyramid. If the circle in plan be divided into any convenient number of parts and the points of division be joined to the centre, these lines will represent the plans of as many sloping lines upon the cone. Their elevations will be cut by the line of section and the plans of the points can be determined by projectors as in fig. H.

A curve drawn through the plans of the points gives that of the entire section.

Problem XI.

A square block 4" edge by 1" thick rests upon the h. p.; *another square block of 2" edge, 1" thick, stands upon the former over its centre; the similar edges of each solid being parallel. The upper surface of the second block is the base of a right pyramid 3" high. Draw a Plan of the whole and an Elevation upon a v. p., which makes an angle of 20° with any horizontal edge in the group.*

Commence with the plan, which is a square of 4" side having another square of 2" edge within it; the centres of both being coincident, and their sides parallel to each other. Draw the diagonals of the smaller square, and the plan of the group will be complete. Assume a ground line, making 20° with either of the sides of the square, and determine the elevation of the lower plinth, remembering that the height of the perpendicular edges is 1". Above this draw the elevation of the second plinth and the axis of the pyramid.

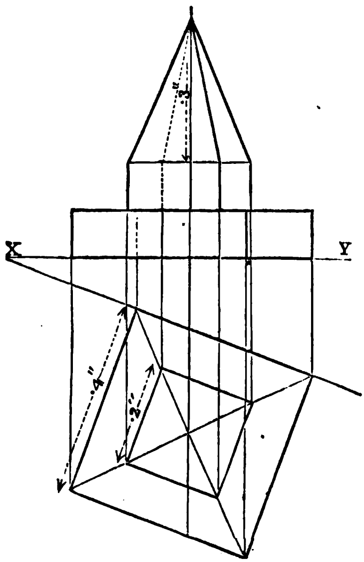


Fig. 102.

Mark off 3" upon the latter, measur-

ing from the highest surface of the second block, and the elevation of the apex of the pyramid will be obtained. Join this point to the four corners of the top surface of the smaller plinth, and the elevation will be complete.

EXERCISES.

1. Draw plan and elevation of a cube of 2″ edge, when its base is horizontal and ·5″ above the paper; its horizontal edges making angles of 30° with the vertical plane.
2. Draw plan and elevation of a square prism (size at pleasure), when its long edges are horizontal, and one of its faces makes an angle of 27° with the paper.
3. Draw a plan and elevation of a cube, with a square prism standing upon it, the axes of both solids forming one straight line, the horizontal edges of the prism to make angles of 45° with those of the cube. The elevation is to be drawn upon a vertical plane making an angle of 30° with one face of the cube.
4. Draw plan and elevation of a tetrahedron of 1·5″ edge when its axis is vertical.
5. Show, by its projections, an octahedron when its axis is horizontal and perpendicular to the vertical plane.
6. A cone, base 1″ radius, 3″ high, is cut by a plane at 70° with the axis; the centre of the section being 2″ above the base. Show the plan of the cut.
7. A rectangular box, 4″ by 3″ and 1″ high, supports a cube of 2″ edge upon it, the sides of the latter being parallel to those of the former, and the axes of both being vertical and in the same straight line. A cone stands upon the cube over its centre, base 1″ radius, 3″ high. Draw a plan of the whole and an elevation upon a vertical plane, making an angle of 30° with a face of the box.
8. A circular slab, 2″ radius, 1″ thick, supports upon its upper surface and over its centre a cylinder, base 1″ radius, and 3″ high. Draw plan and elevation, and show upon the plan the projection of a section indicated upon the elevation, by a straight line joining the top left-hand point of the cylinder to the bottom right-hand point of the slab.

CHAPTER III.

ON ALTERATION OF THE GROUND LINE.

In our commencing chapter we noticed that several elevations could be projected from one plan. Thus, an end view of the instrument-box could be obtained by assuming a vertical plane parallel to that end, or an angular view by arranging our ground line accordingly.

In this chapter it is our intention to carry this principle further, and to determine several *plans* from one *elevation*, and *vice versa*.

Let us revert to our illustation of the box upon the table. If it be slightly inclined, it is easily seen that its plan will alter in shape. And if, instead of moving the box, we could suppose the table to be so tilted that its surface should make, with the base of the former, an angle equal to that which existed under the first conditions, the projection of the box upon the plane of the table would then be exactly the same shape as before.

Now, the removal of either of the co-ordinate planes is effected very easily by an alteration of the ground line. And, when it is desired that an object should have either of its surfaces inclined to one of the co-ordinate planes, the most ready means of effecting this is to project the solid first upon planes parallel to the surface in question, and afterwards, by the removal of X Y to assume a fresh plane, upon which the desired drawing can be made.

The consideration of the problems which follow will make this principle clear.

PROBLEM XII.

A Hexagonal Prism has its axis inclined 40° to the paper, and one face parallel to the v. p. *Draw plan and elevation.*

Draw the hexagon with one side parallel to X Y, which is the plan of the solid when standing with its base upon the paper. And by arranging the figure in this way the student will see that one face of the object will be parallel to the v. p.

The elevation must be deduced from this plan, as described in the preceding chapter.

Upon the elevation draw K′ K, to represent the axis of the solid and produce it.

If a fresh X Y be now assumed, making an angle of 40° with the line K′ K, the elevation will then be that of the solid, with its axis inclined as regards the new h. p. The best way for the student to grasp this principle is to fold his paper upon the new X Y, so as to show a h. p. and a v. p. He will then see his first elevation under quite a different aspect. It will be that of a solid tilted over.

To determine the plan, projectors must be drawn through every point of the elevation, perpendicular to the assumed X Y, and lengths must be measured along each of these projectors, equal to the distances of the points in the first plan, from the first ground line (fig. 103).

Thus, taking the point A for an example, a projector a' 1 passes through a' perpendicular to the new X′ Y′. The distance $a'\,a$ is transferred along this projector to the point a_1 beyond the ground line; that is, $a'\,a$ is equal to $1a_1$.

All the other points of the base are projected in the same way, and a new plan of it is thus obtained.

The plan of the other end of the solid is similarly determined by projectors through its points in elevation.

And as the first plan is that of both ends, the distances to be measured along the projectors first drawn will be exactly the same as before.

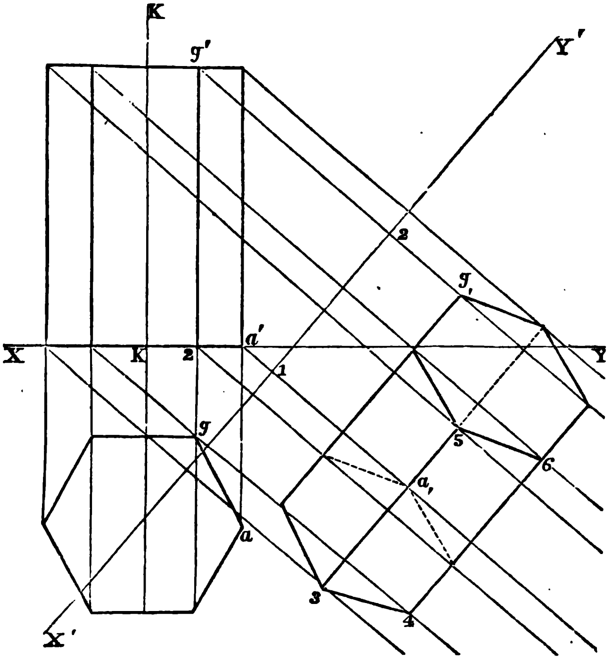


Fig. 103.

To illustrate this, take point g'. The distance measured upon the projector beyond X′ Y′ is equal to 2 g.

The whole plan is completed by joining the similar points in each base, as shown in the diagram.

It should be noticed that lines which are parallel in the solid are still parallel, however they may be projected, thus 3·4 is parallel to 5·6.

A little consideration of the position of the solid

will show that that part of the base in which *a* is situated is hidden, and that the opposite end is wholly seen in plan. The edges dotted will indicate this.

Problem XIII.

To draw the projections of a Hexagonal Pyramid when one of its triangular faces is, (1*st*), *horizontal, and* (2*d*), *vertical.*

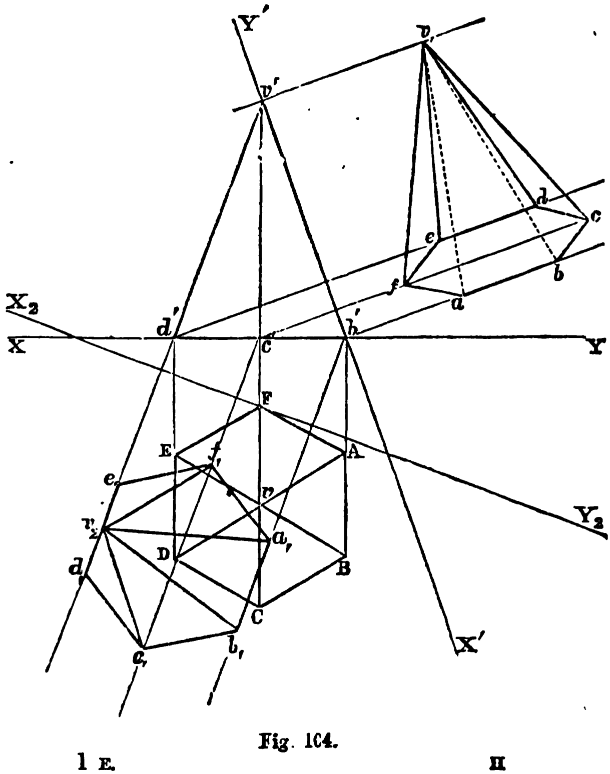


Fig. 104.

Commence with a plan and elevation of the solid, when standing with its base upon the paper, one side of the plan being perpendicular to X Y. It will be seen then, that two of the faces will be represented in the elevation by the lines $b'\ v'$ and $d'\ v'$. If an X Y be assumed containing one of these lines as $b'\ v'$, the elevation, with regard to the new h. p., will be that of the solid, when its triangular face, A B V, is horizontal, in fact when it is in that plane. A fresh plan of the base can then be determined by projectors perpendicular to the new X′ Y′; the distances of the points in plan from the first v. p. being transferred to the new plan. It should be noticed that the projector, $b'\ a\ b$, has two points upon it at distances from X′ Y′ equal to b' A and b' B respectively.

The apex of the pyramid must be projected in an exactly similar manner and $v'\ v$ must be made equal to $c'\ v$.

In the diagram a plan of the solid is shown when the opposite triangular face is vertical. A new $X_2\ Y_2$ is in that case drawn perpendicular to the line $d'\ v'$. Under those circumstances the plan of the vertex falls in the centre of the plan of one edge of the base.

PROBLEM XIV.

To draw the plan and elevation of a Hexagonal Pyramid when one of the edges meeting in the vertex is (1*st*), *horizontal, or* (2*d*), *vertical.*

The solid, as before, must be drawn when resting with its base upon the paper, but it must have one of its diameters or sides parallel to X Y. The effect of putting the plan in this position will be that two opposite edges of the solid will be parallel to the v. p., and consequently their elevations will be as long as the edges themselves. And whenever a line in a solid is to be inclined and fresh plans or elevations of it determined, it is necessary,

in the first place, that the object be so drawn that the line in question and one of its projections shall be equal in length.

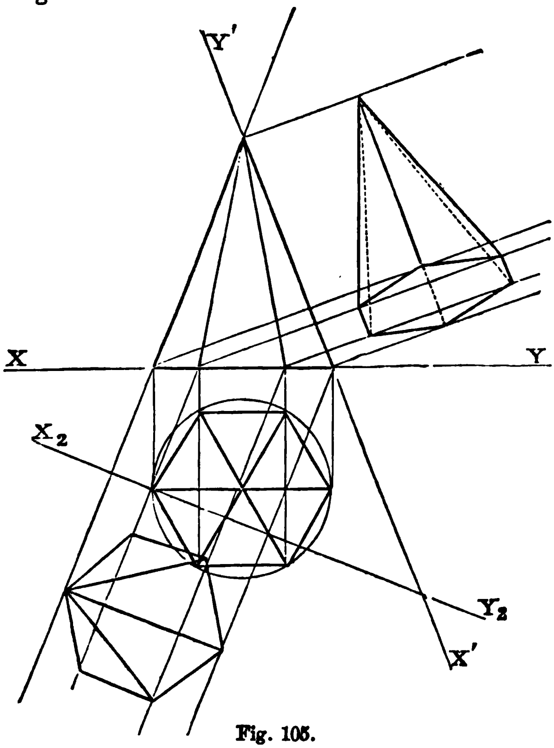


Fig. 105.

A new h. p. can then be assumed by taking X' Y' or X_2 Y_2, parallel or perpendicular to one of these edges, according as that line is to be horizontal or vertical.

The determination of the respective plans upon these planes is obtained by the same construction as in the preceding problem.

Problem XV.

To draw a plan and elevation of a Square Prism, when the diagonal of the solid is (1*st*), *horizontal, or* (2*d*), *vertical.*

It was shown in the preceding problem that when a line in a solid is to be inclined, and its projections

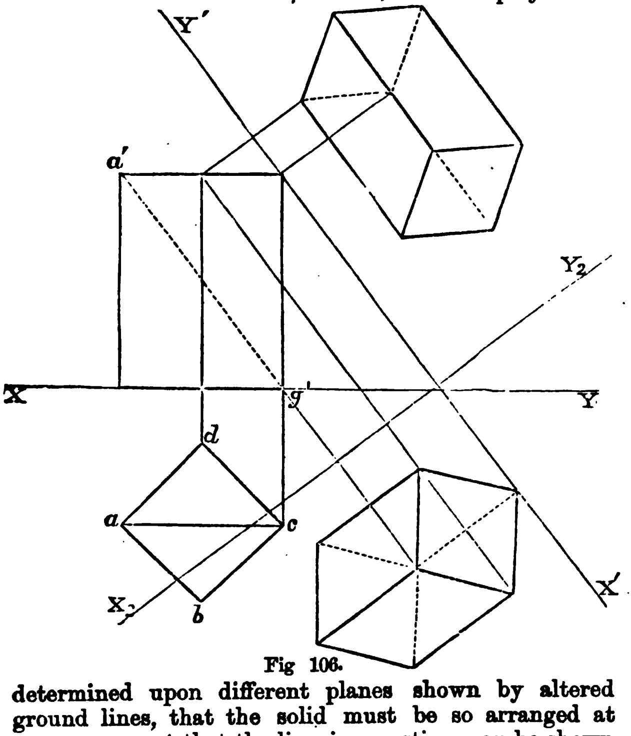


Fig 106.

determined upon different planes shown by altered ground lines, that the solid must be so arranged at commencement that the line in question may be shown

its full length in either plan or elevation, and that this is an invariable rule.

In the problem before us it is necessary so to place the prism that its diagonal may be parallel to the v. p.

The square, *a b c d*, which is first drawn, must therefore have its diagonal parallel to X Y. When the elevation is complete, the line *a′ g′* will represent that diagonal which is to be horizontal or vertical.

A parallel, X′ Y′, to it must be assumed as a ground line, and the plan of the solid determined as before. This will satisfy the first condition in the question.

The second part of the problem is solved by assuming the h. p. perpendicular to the diagonal; that is, by taking X_2 Y_2 at a right angle with *a′ g′*, and proceeding as before. It should be noticed that in the latter case two points coincide in plan, in fact that one point is the projection upon a horizontal plane of a vertical line.

PROBLEM XVI.

To determine the plan and elevation of an Octahedron when it rests with one of its triangular faces upon the paper.

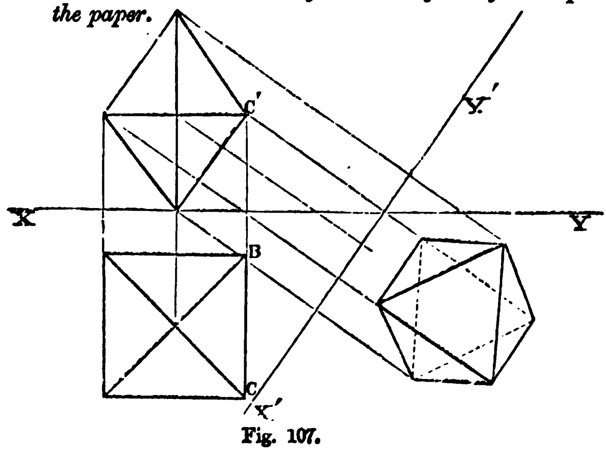


Fig. 107.

Commence with a plan of the solid when its axis is vertical and one of its edges perpendicular to X Y. We thus ensure that the elevations of the triangular faces which have B C for their common base are straight lines.

Assume an X′ Y′ parallel to either of these, as in fig. 107, and determine the plan as before.

The boundary line of an octahedron in this position is a hexagon.

Problem XVII.

Given the projections of any Solid; to determine other projections from them.

Let the figure $a\ c_1$ be the plan of a square prism, of which A′ B′ C′ D′ is the end elevation. And let a new

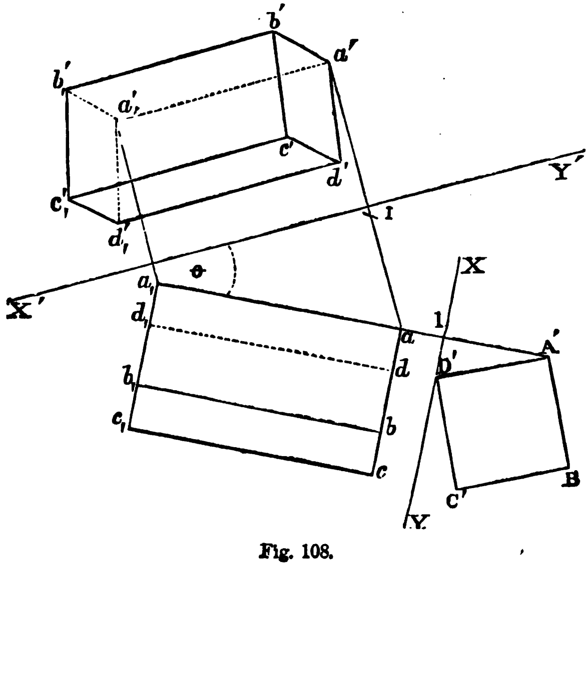


Fig. 108.

elevation be required upon a v. p., making an angle, θ, with the long edges of the solid.

Assume X′ Y′, making the required angle with either of the plans of the sides, as $a\ a_1$.

Projectors through *a b c* and *d* in the plan, perpendicular to X′ Y′, will contain the required elevations of the points A B C D. Now, as the heights of these points above the h. p. are shown in the given end elevation, it is only necessary to transfer them from one elevation to the other. The distance, $1a'$, in both cases is the same.

The elevation of the end, A, B, C, D, is obtained in a similar manner; and as the solid is lying horizontally upon one of its sides, the heights of the corners are the same as those of *a b c d*.

Problem XVIII.

An Irregular Pyramid has for its base a triangle, A B C. A B = 3″; A C = 3·5″; B C = 4″; *the plan, d, of the fourth corner projected upon the plane of the base, is* 2″ *from* A *and* 1·5″ *from* B. *The true length of the remaining edge,* C D, *is* 3·7″. *Draw the plan of the Pyramid when standing on its base, and an elevation on a plane parallel to the edge* A D. *Determine the length of the edge* B D, *and the inclination of the face* A B D.

The whole of this question turns upon the proper arrangement of the ground line.

Draw a triangle, A B C, having its sides equal to those given in the question. With A as centre, radius 2″, describe an arc, and with B as centre, radius 3·5″, describe another arc, intersecting the former in *d*, which is the plan of the apex, D, of the pyramid.

Join *d* A, *d* B, and *d* C to complete the whole plan. The height of the apex, D, can be obtained by a similar construction to that employed for determining the

height of a tetrahedron (Chap. II). At d in C d,

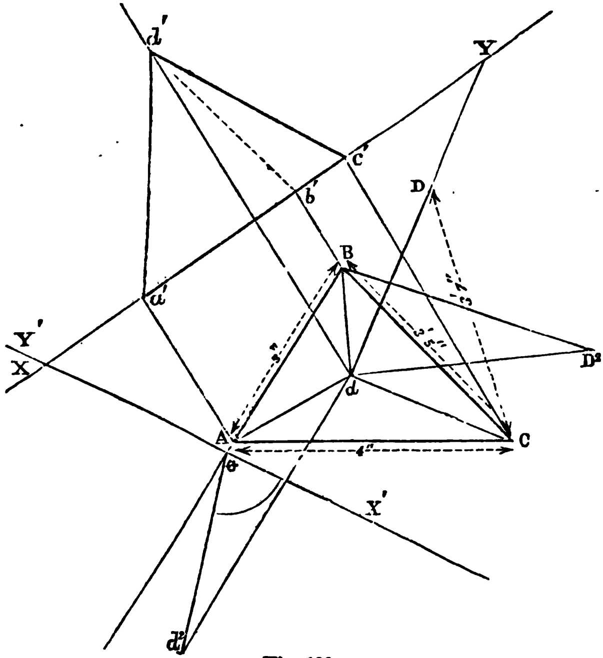


Fig. 109.

draw d D perpendicular to it, and with C as centre, radius 3·7" (the real length of C D), describe an arc, intersecting d D in D. Then d D is the height of the pyramid.

The elevation upon a plane, parallel to the edge A D will present no difficulty.

To determine the length of the edge B D, set out a perpendicular to B d at the point d, making d D^2 equal to the height of the pyramid, as obtained above. Then B D^2 is the true length required, and D^2 B d is its inclination.

If a ground line be assumed perpendicular to the edge A B, and an elevation be made upon the new v. p., the whole face, A B D, will be shown as one line, and the angle it makes with X Y is the inclination of that face to the paper.

Problem XIX.

Given, the line 1′ 3′ *as the elevation of a section of a Square Prism* (*fig.* 110); *to determine its true shape.*

Assume the line 1′ 3′ to be the ground line of a plane upon which the plan of the figure 1 2 3 4 is to be projected. Set out projectors from each of the points 1′ 2′ 3′ 4′ perpendicular to the line 1′ 3′ and measure lengths equal to the distances of *a b c* and *d* from X Y. Then, by joining the points found, the true shape of the section will be determined.

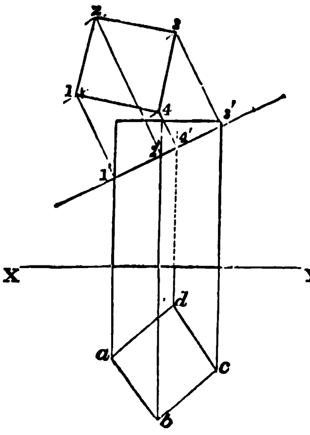


Fig. 110.

Problem XX.

Draw the plan of a Right Pyramid, whose base is a hexagon of 1·25″ *side, and its axis* 3·25″, *when it stands upright upon a horizontal plane. Give the section by a vertical plane which cuts off half of one edge and a quarter of the next, measuring downwards from the vertex.*

When the projections of the pyramid, under the con-

ditions given in the question, have been determined, draw a line of section upon the plan, bisecting $v\ c$ in g, and passing through h, $v\ h$ being $\frac{1}{4}$ of $v\ b$.

The line of section is drawn upon the plan, because the cutting plane is a vertical one. A moment's consideration will tell us that, if a cutting plane be perpendicular to either of the co-ordinate planes, the projection of the section will be a straight line upon that plane. Thus a horizontal section would be shown by a horizontal line upon an elevation.

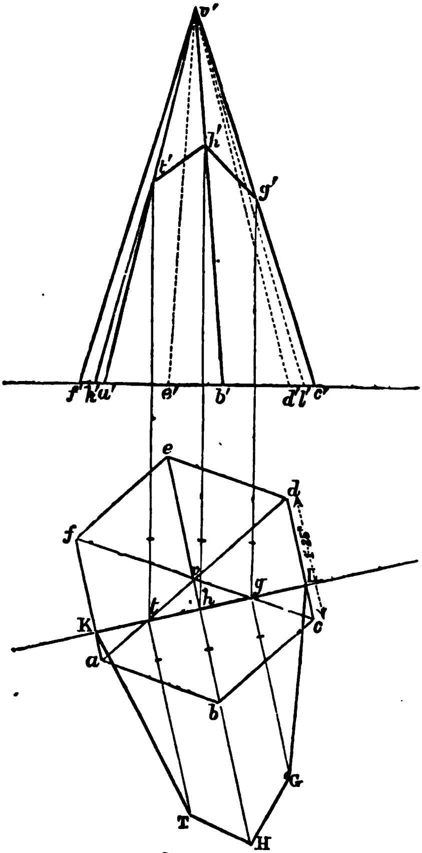


Fig. 111.

K L drawn upon the plan, as explained above, shows that two edges, A F and C D, of the base would be cut. Determine the elevation of the section by projectors through the points in plan meeting the eleva-

tions of the lines upon which they occur. Two of these, k' and l', will be upon X Y. The other three, l' g' and t', are found in the usual manner.

The figure k' t' h' g' l' will be the elevation of the section.

The true shape is determined when the five points of the section are constructed into the h. p. upon the line K L as an axis.

Those two will remain stationary, and the others, G H and T, will fall in perpendiculars to k l, passing through g h and t. The distances h H, g G, and t T must be made equal to the heights of H, G, and T above the h. p., as shown in the elevation. Then the figure, K T H G L, will be the required true shape of the section.

Problem XXI.

If a Solid be cut by a plane of section, and a plan or elevation of one of the parts of it be made upon that plane, such a drawing is called a Sectional *Plan or Elevation.*

The projections of a cube are shown in fig. 112, and the line S T represents on the plan a section of the solid made by a vertical plane. The solid is thus divided into two unequal parts. It is required to make a sectional elevation of that portion furthest from the v. p.

Commence by determining the elevation of the figure made by the section on that of the cube. The student will find it very advisable to "letter" both plan and elevation very carefully, so that he may not be confused in determining which edges will be cut.

Take one or two points as examples. The plan shows that the point 1 in the section occurs upon the edge E H; a projector, therefore, through the point 1 in e h, meeting e' h' in 1', will give the elevation of that

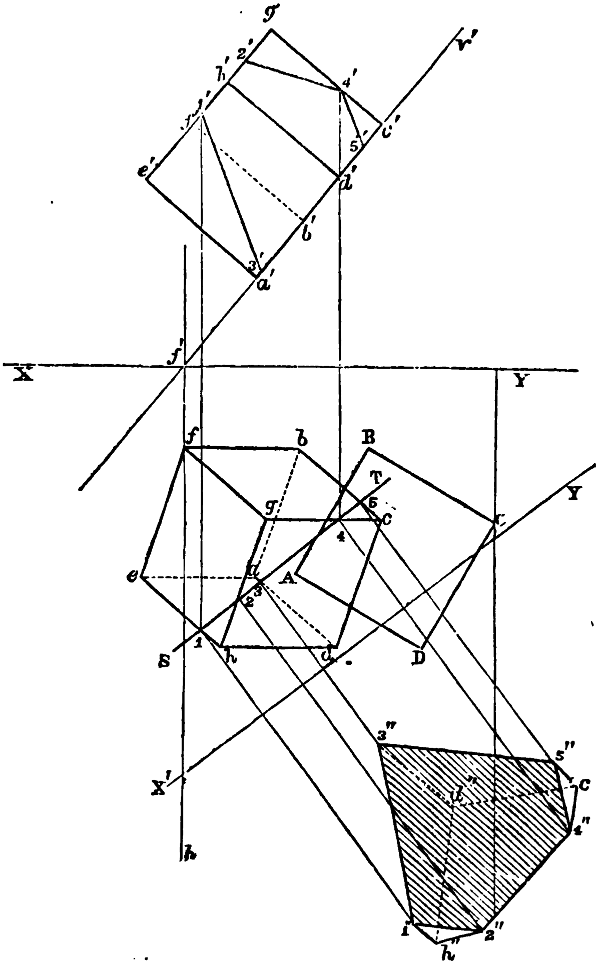


Fig 112.

point. Again, point 4 occurs upon the edge G C. Determine its elevation therefore upon $g' c'$.

Having shown the section upon the given elevation, we know the heights of all the points of that section. We then assume a new ground line parallel to S T,* and determine $1'' \; 3'' \; 5' \; 4'' \; 2''$ its true shape.

A further reference to the plan shows that the only other points of the solid which exist in the portion cut off by the plane of section, are H D and C. We must project these points upon the plane of section in the usual manner (their distances from $X_{\prime} \; Y_{\prime}$ being equal to those in the first elevation from X Y).

To discover which points are to be joined, we must again refer to our plan. There we find C is joined to 4 and 5; D to 3 and C and H; and H to 1 and 2.

EXERCISES.

1. Draw plan and elevation of a square pyramid—base 1" side, height 3", when one of its long edges is inclined 20° to the paper.

2. A pyramid 3" high has a pentagon A B C D E of 2" side for its base. Show this solid by *one* elevation and *two* plans.

 When the two edges B V, C V are *horizontal.*

 When the edge A V is *vertical.*

3. A cylinder 4" long, its base being a circle of 1·5" radius, to be drawn in plan and elevation in one of the following positions:—

 a. Its axis inclined to the paper at 35°.

 b. When a vertical plane touches the two ends in opposite points of parallel diameters.

4. A hexagonal prism, base 1" edge, and 3" long, has its axis horizontal, one of its faces being inclined 12° to the paper. Draw plan and elevation, and a second elevation upon a vertical plane, making an angle of 40° with the plan of the axis.

5. A cube, 2" edge, rests upon a circular slab, radius 3", 1" thick. The centre of the cube is over the centre of the slab. Draw plan and elevation of the whole, when a corner of the cube and an edge of the slab rest upon the paper.

6. Draw plan and elevation of a tetrahedron, 2" edge.

 (1.) When one of its faces is vertical.

 (2.) When one edge is vertical.

 (3.) When that edge is horizontal.

* The ground line is taken parallel to S T, so that the drawings may be clear of each other. The result would be the same if S T itself was assumed as ground line.

7. A pyramid having for its *base* a square 2·5″ side, and its axis 3·25″ long, rests with one *face* on the horizontal plane. Draw its plan, and a sectional elevation on a vertical plane represented by a line bisecting the plan of the axis, and making an angle of 60° with it.

8. An hexagonal pyramid (side of base 1·5″, axis 4″) has one edge of its base horizontal, and the plane of that base inclined at 60°. Draw the plan, and show the real form of the section made by a horizontal plane bisecting the axis when the solid is so inclined.

CHAPTER IV.

ON OBLIQUE PLANES.

If the surface of a solid is not parallel to either of the co-ordinate planes, it is said to be *inclined* to that plane.

Thus, the lid of a mathematical instrument box, when it is slightly open, presents a surface inclined to the horizontal; whilst a shutter hung upon hinges, when partly open, is an instance of a surface inclined to the vertical plane of the window. If the surface of the lid could be produced until it met the ground, it would make by its intersection therewith a line. In the same way, the line upon which the shutter revolves, would represent the intersection of its inclined surface with the vertical plane of the window.

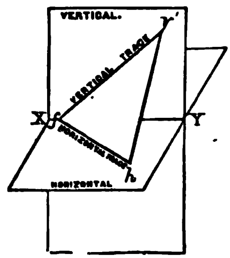


Fig. 113.

It is by means of the lines in which an inclined plane would intersect the co-ordinate planes that the position of such surfaces is indicated.

If a piece of cardboard be cut in the shape of a triangle, and it be fitted into one of the angles of the model mentioned in Chap. I., *Solid Geometry*, as shown in fig. 113, its edges will mark two lines, $v'f$ and fh, upon the vertical and horizontal planes, meeting in a point f in X Y. These lines would be called the *traces* of the oblique plane. The trace $v'f$ upon the v. p. would be called the vertical trace (v. t.), and fh, that upon the horizontal plane,

the horizontal trace (h. t.). When the co-ordinate planes are made to coincide by the revolution of the v. p. upon X Y, these traces will be represented as making a larger angle with each other than they actually do upon the oblique surface.

In all cases but *one*, the traces of oblique planes meet upon X Y.

In fig. 114, traces of planes in different positions are shown.

The first plane (fig. 114 *a*) is inclined to the h. p., but is perpendicular to the v. p.

The second (fig. 114 *b*) is the reverse of the above, as it is inclined to the v. p., but perpendicular to the h. p.

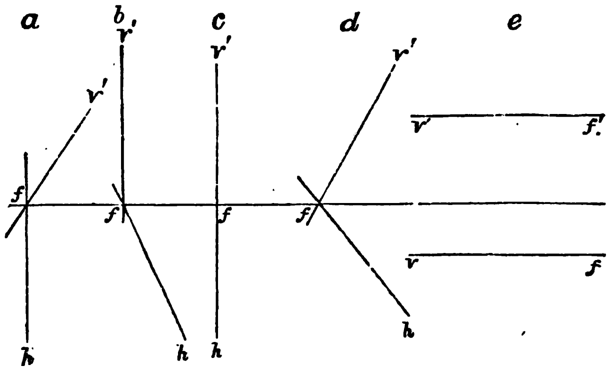


Fig. 114.

The third (fig. 114 *c*) is perpendicular to both the co-ordinate planes.

The fourth (fig. 114 *d*) is inclined to each of the co-ordinate planes.

The fifth plane (fig. 114 *e*) is inclined to both the co-ordinate planes, but is parallel to X Y.

As planes are infinite, so their traces are infinite, and must not be supposed to terminate upon X Y, in fact,

it is advisable in all cases to draw these lines a short distance beyond the ground line.

The number of different positions in which the traces may occur is infinite, as any two lines meeting in X Y may represent an oblique plane.

Referring again to figure 114 *a*, the angle which the line *v' f* makes with X Y is the inclination of that plane to the horizontal.

Similarly, the angle which *f h*, in fig. 114 *b*, makes with X Y shows the inclination of that plane to the vertical.

Neither of the lines *v' f'*, *f h*, in fig. 114 *d*, will tell us the inclination of that plane. To determine this a special construction is necessary, which will be explained hereafter.

All lines which lie entirely upon a plane are said to be contained by it. If a pencil be laid upon an oblique surface in such a manner as to be parallel to its h. t., it will be seen that the pencil is horizontal. Again, if the pencil be placed perpendicular to the h. t., and lying in the plane, it will be inclined to the horizontal fully as much as the plane itself. Therefore, in any other position upon this plane, the pencil would be less inclined than the oblique surface.

If two oblique planes be parallel to each other their traces will be parallel.

An oblique plane is said to be "constructed" into either of the co-ordinate planes when its whole surface, with the points and figures upon it, is made to revolve upon one of its traces until it coincides with that plane.

When the inclination of a plane is given it must be understood to mean its inclination with the h. p., without it be otherwise stated.

Problem XXII.

Given, an Oblique Plane, by its traces, to determine its inclination to both planes of projection.

The angle of inclination of an oblique surface to the

h. p. can be measured by finding the cone which, when its base is horizontal, is tangent to that surface.

This is easily illustrated by taking a model of a cone and placing a piece of paper against its side. Then, whatever the slope of the side of the cone may be, the inclination of the surface of the paper will be the same. If the paper be made to have one of its edges upon the ground, that edge will represent the h. t. of an oblique plane. It will easily be seen that under these circumstances the paper touches the cone in a line which passes through the apex, and which is perpendicular to the h. t.

Assume, therefore, a point, a', in the v. t. $v' f'$ of the given plane and consider it to be the apex of a cone. The line $a' a$ perpendicular to X Y will represent its axis, and a circle described with A as centre, and tangent to the h. t. $f\ h$, its base. Only part of this circle need be drawn, as in the diagram. The point b, where the base of the cone meets the h. t., is the foot

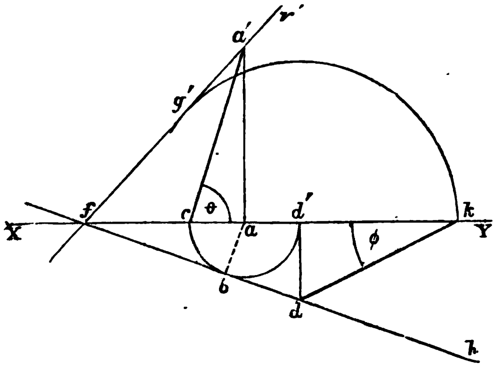


Fig. 115.

of the line of contact of cone and plane, the other extremity of that line being in the apex A.

Join c to A′, and the angle which A′ c makes with

X Y is the base-angle of the cone, and also the required inclination of the plane.

Note.—It is not necessary in solving such problems as these to give the actual number of degrees in the inclination.

It is usual to describe a small arc in the angle found, and to place either the mark θ or Φ, according as the inclination is to the horizontal or vertical plane.

To determine the inclination to the v. p., the same kind of construction is adopted; but in this case the base of the cone must be in the v. p., and its axis must be horizontal, instead of vertical.

Assume any point, *d*, in *f h*, and draw *d d′* perpendicular to X Y. This is the plan of the axis of the cone. With *d′* as centre, describe arc *g′ k* tangent to *v′ f*, and meeting X Y in *k*. Join *k d*, and the angle marked Φ is the required inclination to the v. p.

A second example is shown in figure 116, in which a modification of the above construction is necessary. The oblique plane, *l′ m n*, is in such a position that the cones which measure its inclinations must be fitted to it, beneath the h. p. or behind the v. p.

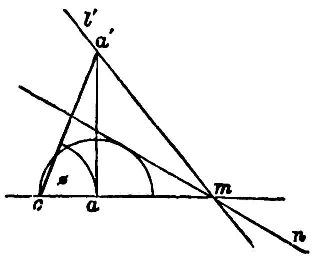


Fig. 116.

Assume, as before, a point *a′* in the v. t., and draw *a′ a*. The arc of the circle must then be described above X Y, tangent to the h. t. produced. The point *c* being joined to *a′*, the angle *a′ c a* is determined as before.

The determination of the inclination to the v. p. will present no special difficulty.

PROBLEM XXIII.

To determine the inclination to both planes of projection of an Oblique Plane, having its traces parallel to X Y.

A right angled triangle can be conceived as standing perpendicular to the v. p., and so fitting under the oblique one, that the hypotenuse of the triangle may be contained by it. The acute angles of such a triangle would then present the required inclinations.

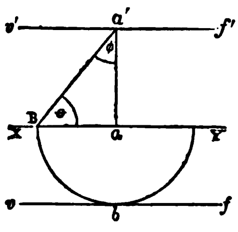


Fig 117.

Take any point *a'* in *v' f'*, and draw *a' b* perpendicular to X Y, intersecting that line and h. t. in points *a* and *b*. With *a* as centre, and *a b* as radius, describe the arc *b* B, and join *a'* B. Then the angle θ is the inclination to the h. p., and ϕ is that to the v. p.

Note.—This is really only a repetition of the construction in the preceding problem, as a cone is generated by the revolution of a right-angled triangle upon its perpendicular. But the writer has found this case better understood by beginners as treated above, than by adopting the idea of a tangent cone, as in Problem XXII.

PROBLEM XXIV.

Given, either of the traces of an Oblique Plane and its inclination to the h. p., *to determine the remaining trace.*

Referring to figure 115, let *f h* be the given h. t., and θ the given inclination.

Take any point *a* in X Y as centre, and describe an arc *b c* tangent to *f h*, meeting X Y in *c*. At *a* erect an indefinite perpendicular *a' a*, and at *c* make *c a'*, making an angle with X Y equal to the given in-

clination, and meeting the perpendicular in *a′*. Join *a′ f*, and *v′ f* will be the given trace required.

If *v′ f′* be given instead of *f h*, and it is required to find the latter, assume a point *c* in X Y, and make *c a′* at an angle with it equal to the given inclination, meeting the v. t. in *a′*. Draw *a′ a* perpendicular to X Y, and with *a* as centre, radius *a c*, describe the arc *a c b*. Draw *f h* tangent to this arc, and it will be the h. t. required.

Problem XXV.

To determine the traces of a Plane inclined 60° *to the* h. p., *and* 40° *to the* v. p.

This is considered an *advanced* Problem. It is inserted here to make the series of questions upon traces of oblique planes complete. The student may, without any inconvenience, defer its study to a later part of the course.

To solve the above, it is necessary to determine two cones, whose generatrices* shall make the given angles of inclination with the co-ordinate planes. These two cones must have their axes *in* those planes, meeting in one point upon X Y. They must also envelop a common sphere, having its centre in X Y, at the point where the axes of the cones meet. Then the plane which touches both these cones is that required in the question.

Draw a line 1·2 perpendicular to X Y, and at any point, *c*, make an angle of 60°. The line *c a′* meets the perpendicular 1·2 in *a′*. Then *a′ c a* will be the elevation of half a vertical cone. Describe the arc *c b* to represent part of the base of that cone in plan. Then, with *a* as centre, describe a circle *e g n*, which shall

* The hypotenuse of the right angled triangle which generates a conical surface is the generatrix of the sloping surface thereof.

be tangent to the line a' c. This is the elevation of the sphere enveloped by the cone.

Make the line a_2 k tangent to the circle $e\,f\,g$, and meeting X Y at an angle of 42°. Then the triangle a_2 K a will be the plan of the horizontal cone, also enveloping the sphere.

With a as centre, a k as radius, describe the arc k s t, to represent the elevation of part of this cone. Then v' f, drawn through the point a' and touching the

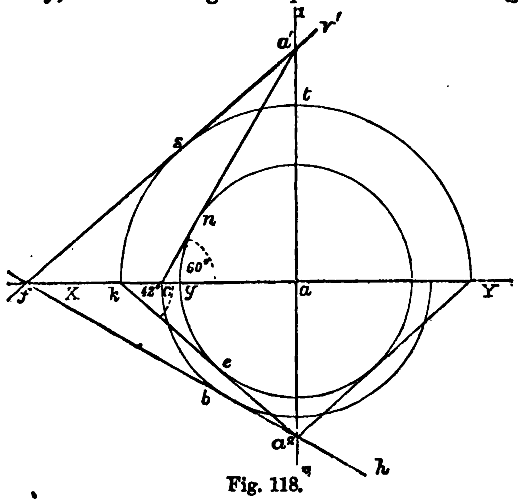


Fig. 118.

arc, k s t, in the point s, will be the required v. t.; and f h passing through f and a_2, will be the required h. t.

Note.—The sum of the inclinations of an oblique plane must be between 90° and 180°.

Problem XXVI.

To determine the distance between two parallel Planes given by their traces.

Let v' f h and l' m n be the traces of the given planes. Proceed as if to find the inclination of the plane l' m n. Produce a a' beyond a' to meet v' f in a_2, and with a as centre, describe an arc tangent to f h, meeting X Y in b. Join b a_2, and the perpendicular distance between the lines b a_2 and c a' is that between the two given planes.

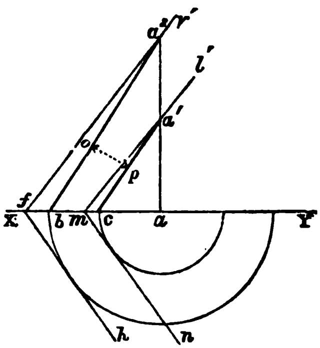


Fig. 119.

Problem XXVII.

To determine by its traces a Plane parallel to a given plane, and at a given distance from it.

This is the converse of the preceding problem; and, referring to fig. 119, let the given plane be v' f h, and o p the distance. Proceed as if to find the inclination of v' f h, and at a perpendicular distance, equal to o p, draw c a' parallel to b a_2. Then a' is one point in the vertical trace of the required plane; and as parallel planes have parallel traces, l' m and m n, drawn parallel to v' f and f h respectively, will be those of the plane required in the question.

Problem XXVIII.

To determine the Angle which exists between the traces of an oblique plane, when the co-ordinate planes are in their proper position.

If the inclined plane be "constructed" into the h. p., the v. t. will, when drawn upon the paper in its new

position, make the angle required with the h. t. Imagine, once more, the cone fitting under the plane, and conceive the position of the line of contact after the "construction" into the h. p. has taken place. It will be perpendicular to the h. t. Draw, therefore, any line, $a\ b$ (fig. 120), perpendicular to $g\ h$. Determine $a'\ c$, as in Problem XXII. Produce $a\ b$ beyond b, making b A equal to $c\ a'$. Join f A, and $h\ f$ A is the angle between the traces. The method of proof of this construction would be, to cut a piece of paper so that it may have one angle equal to $h\ f$ A, and to try if it would fit the traces given, when the horizontal and vertical planes are mutually perpendicular.

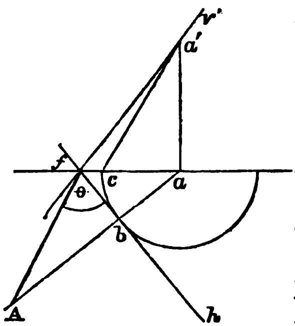


Fig. 120.

Problem XXIX.

Given, the plan a of a point, A, which is contained by the oblique plane, v' f h, to determine its elevation.

Four examples of this problem are shown in fig. 121. In the first case, as the plane is perpendicular to the v. p., the elevation of the point will fall upon the v. t., and its position, a', is determined by a projector from a. In the second case, a line, $a\ b$, must be drawn parallel to the h. t. This line contains the plans of all the points which lie upon the plane, $v'\ f\ h$, and are at the same height above the h. p. as A. At the point b, raise the perpendicular $b\ b'$, and the length of this line is the height of the point A above the paper. Through b' draw $a'\ b'$

parallel to X Y, and the point *a'*, where this parallel intersects a projector from *a*, is the elevation required. The third case is only a modification of the second, and

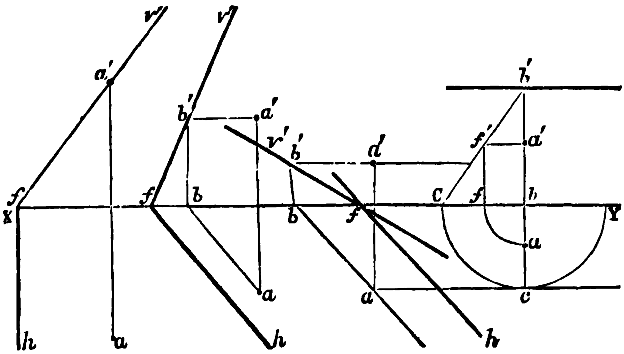


Fig. 121.

reference to the figure will be sufficient to explain its solution.

The fourth case requires a special construction. Draw the line *b' c* perpendicular to X Y, and passing through the point *a*. With *b* as centre, describe the arcs *c* C and *a f*. Join C *b'*, and at the point *f*, draw *f f'* perpendicular to X Y, meeting C *b'* in *f'*. Through *f'*, draw *f' a'* parallel to X Y, and the point *a'*, where this line intersects *b' c*, is the elevation of A.

A little consideration, with the use of a model of the plane in the given position, will show the principle of the above construction.

b C *b* is a right-angled triangle, which, in its original

position, was perpendicular to the v. p., and contained the point A upon its hypotenuse. It is here shown constructed into the v. p., and the perpendicular $f f'$ indicates its height above the h. p.

Note.—If the elevation of the point be given, and the plan required, the construction would be the converse of the above.

Problem XXX.

To determine the traces of a Plane, parallel to a given one, and passing through a given point.

Let $v' f h$ be the given plane, and $a' a$ the projections of a point, A. Through a draw $a b$ parallel to $f h$, and raise $b b'$ perpendicular to X Y. The point b', where a parallel ($a' b'$) to X Y meets this perpendicular, is in the vertical trace of the required plane. Its traces being parallel to those of $v' f h$, draw $l' m$ and $m' n$, as shown in the diagram.

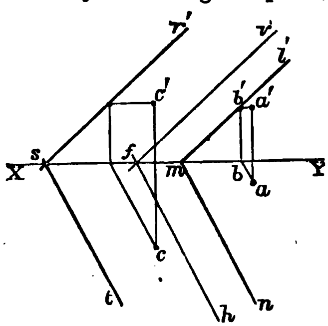


Fig. 122.

A second plane, $r' s t$, is shown in the figure passing through the given point C. The construction is similar, and needs no detailed explanation.

Problem XXXI.

To determine a Line perpendicular to a given plane (which is inclined to h. p., *but at a right angle to the* v. p.), *to pass through a given point,* P.

The plan and elevation of a line which is perpendicular to an oblique plane, are perpendicular to the traces of that plane—the plan to the h. t., and the elevation to the v. t. This is easily proved by holding

a pencil perpendicularly to any inclined surface, and viewing it in two directions, at a right angle with each other. Through the given projections of the point, draw $p' r'$ and $p\ r$ perpendicular to the traces of the plane. The plan of the extremity, R, is determined by a projector through r'.

The given plane being inclined only to the h. p., the elevation of lines perpendicular to it will be fully as long as those lines themselves.

Perpendiculars to planes, which are inclined to both the horizontal and vertical planes, will be treated in a future chapter.

PROBLEM XXXII.

To determine, by its traces the Plane containing threc given points.

Let $a' b' c'$, $a\ b\ c$ be the projections of the given points, it is required to determine a plane which shall contain them.

If two points are contained by a plane, it is clear that the line joining those two points must also be contained by that plane. Also, if a line be contained by a plane, the traces of that line are in the traces of the plane. These two principles are sufficient to solve this problem; for the plane required must contain each of the three lines, A B, B C, A C, the traces of which will be points in the traces of the plane.

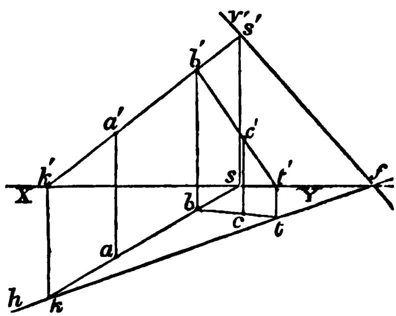


Fig. 123.

Join $a' b'$, $a\ b$, $b' c'$, $b\ c$, and produce $a' b'$ beyond b' to meet

X Y in k'. Then a perpendicular to X Y through k, intersecting the plan of A B produced in k, gives one point in the required h. t., and t which is the h. t. of the line, B C is a second point in that trace. The line $f\,h$, drawn through these points, is the h. t. of the required plane. Find s, the v. t. of the line A B, and draw $v'f$, passing through s, to meet $f\,h$ in f. Then $v'f\,h$ is the plane containing the three given points.

EXERCISES.

1. Draw any two lines meeting in X Y. Consider them as the traces of an oblique plane. Determine its inclination to the horizontal plane, and show the plan of a horizontal line lying in it 1″ above the horizontal plane.

2. The horizontal and vertical traces of a certain oblique plane, make angles of 30° and 80° respectively with X Y. Assume any point above the ground line as the elevation of a point contained by this plane, and determine its plan.

3. Draw a line parallel to X Y, at a distance of 1″ from it. Consider this as the horizontal trace of a certain plane inclined 40° to the horizontal plane, and determine the vertical trace.

4. The horizontal trace of a plane makes an angle of 30° with X Y; the vertical trace one of 50°. Determine the inclination of the plane and the true angle between the traces.

5. Draw two parallel planes, inclined 60° to the horizontal plane and 1″ apart; their horizontal traces to make angles of 40° with X Y.

6. Assume any three points, A, B, C, by their projections, one of them at least being beneath the horizontal plane, and determine the oblique plane containing them.

7. Draw two parallel lines at angles of 40° with X Y, and above it; draw also two parallel lines from the points where the former intersect the ground line at angles of 30° with it. Consider these lines as the traces of two parallel planes, and determine the distance between them.

8. A triangle 2″ by 1·5″ by 1·8″ is the plan of a figure, the heights of whose corners are 1″, 1·5″, and ·7″ above the paper. Determine the plane containing the triangle.

9. Draw the traces of a plane inclined 70° to the horizontal plane and 35° to the vertical plane.

CHAPTER V.

ON THE PROJECTION OF OBLIQUE SURFACES.

WHEN a figure rests upon or is parallel to either of the co-ordinate planes, its projection upon that plane is the same in shape as the figure itself. But if its surface be inclined to those planes, its projections will differ from it in shape.

This is well illustrated by cutting a square and a circular hole in a piece of cardboard. When it lies upon the paper, the plans of the holes are a square and a circle; but if it be slightly rotated upon one of its edges until the surface is inclined, the plans of the holes upon the original h. p. will be altered in appearance.

The plan of the square may assume the shape of a rectangle, and that of the circle will be an ellipse. In both cases, the projections of the holes are narrower than their originals, and the more the surface of the cardboard is inclined, the greater the loss of width, until, when it stands vertical, the plans of both holes are straight lines.

An alteration, precisely analogous to this, would take place in elevation if the cardboard were inclined to the v. p.

The holes may be considered as forming parts of an oblique plane, when they are situated as above, the h. t. of which would be the edge upon which the cardboard rotated.

When the plane is inclined only to the h. p., the elevations of all lines contained by it are represented by straight lines coinciding with the v. t. The h. t. in such a case is perpendicular to X Y. And it is most convenient to arrange the co-ordinate planes in this position

in relation to *any* oblique surface, for, if other elevations are required, they can be made after one is determined by proper alteration of the ground line.

Each corner of the square hole in the cardboard, when it revolves upon its edge, describes the arc of a circle in its journey, for it is at all times equi-distant from that edge.

And when the v. p. is assumed perpendicular to the oblique one, these arcs can be shown as parts of circles in elevation.

After working a few of the following problems, the student will appreciate the advantage gained by arranging the co-ordinate planes in the manner described above.

Note.—It is necessary to remind the student of a principle demonstrated in Chapter IV. If a line be contained by an oblique plane, and is parallel to its horizontal trace, that line is still horizontal.

Problem XXXIII.

To draw the plan of a Square when its surface is inclined 42°, and one of its sides is horizontal.

As the surface of the square is to be inclined 42°, commence by assuming the traces of a plane inclined at that angle, and rotate the figure from a horizontal position into this plane. The h. t.—*fh* is perpendicular to X Y, and the v. t. makes an angle of 42° with it.

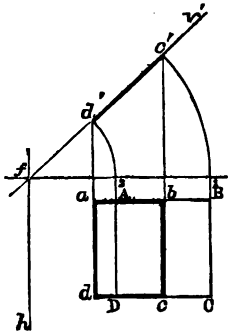


Fig. 124.

The square A B C D must be drawn with one of its sides parallel to *fh*. Then through the corners C and D, projectors must be determined meeting X Y in 1 and 2. With *f* as centre, describe the arcs 1 *c′* and 2 d′, intersecting v. t. in d′ and c′.

These arcs will represent the journey of the points C and D whilst being rotated into the plane $v' f h$. Then c' d' is the elevation of the whole square, because the sides A D and B C of the square being horizontal and perpendicular to the v. p., their elevations are points. The intersections of projectors through d' and c', with lines parallel to X Y through A B C and D, are the plans of the four corners of the square.

Problem XXXIV.

To draw the plan of a Square when its plane is inclined 30°, the diagonal being horizontal.

This problem is only a modification of that preceding it. The plane having been determined by its traces, the square must be drawn upon the paper with its diagonal parallel to the h. t., as that line is to be horizontal. Proceed then, as before, to revolve each of the points, A B C and D into the oblique plane, and thus determine the elevation a' b' c' d'. It should be noticed that c' is the elevation of both the points A and C, as the former is directly behind the latter. Projectors drawn through these points, to meet parallels, to X Y, through A B C D, will give the required plan, *a b c d*. The second diagonal of the square, B D, will be equally inclined with the plane, as it is perpendicular to the h. t.

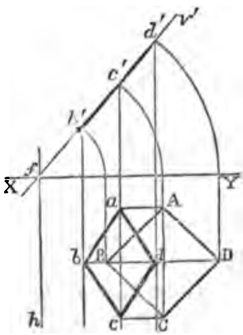


Fig. 125.

PROBLEM XXXV.

A circle has the plane of its surface inclined 50° to the paper; draw plan and elevation.

In projecting curved lines, it is necessary to assume a number of points in those lines, and to determine, individually, the plans and elevations of each of them. When this has been done, a curve passing through the projections determined will give those of the original line, accurate so far as regards the assumed points. The correctness, therefore, of the drawing is augmented, if their number be increased. In the case before us it is most convenient to divide the circle into eight equal parts, so arranged that the diameter A E, which passes through two of them, shall be parallel to X Y. By this means we shall, after that line has been rotated into the oblique plane and its plan determined, have the minor (smaller) axis of the ellipse, which is the plan of the circle. Again, the points H B, G C, and F D, will be respectively opposite to each other, consequently fewer projectors will be required. The remainder of the construction is obvious.

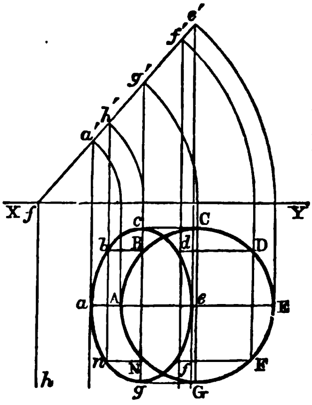


Fig. 126.

The ellipse should be very carefully drawn in by hand.

The elevation of the circle is the straight line $a' e'$ upon the v. t.

PROBLEM XXXVI.

The plan of a Pentagonal Surface is inclined 40°, the line joining two alternate corners of the figure is horizontal; draw plan and elevation.

Commence by drawing the pentagon A B C D E, and

join B D. Consider this line as the h. t. of a plane inclined 40°. Assume an X Y perpendicular to it, and a v. t. making 40° with X Y. The figure must then be revolved upon B D into this oblique plan. The points B and D will not alter in position, and A and E can be treated in the usual way. But the point C, when the figure is thus revolved, will fall below the h. p., as when the portion D E A B is raised, D C B is depressed.

To determine the plan of C, a projector must be drawn through it, and the arc C′ c′ by its intersection with the v. t. will give the elevation of it. The plan is found as before, by a projector through c′ meeting a parallel to X Y through C.

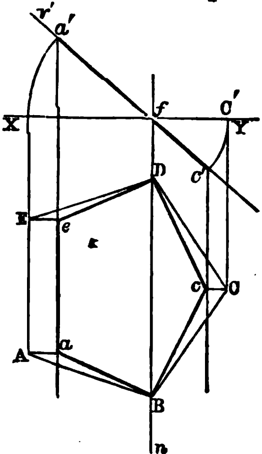


Fig. 127.

Problem XXXVII.

The plane of a Hexagon of 1″ *side is inclined* 25°, *neither of its sides being horizontal; draw plan and elevation.*

After the plane is determined by its traces, the hexagon must be drawn upon the paper in such a position that neither of its sides are parallel to the h. t. The figure must then be rotated into the plane, and the plan determined as before.

Problem XXXVIII.

A Cube has one of its surfaces inclined 50°, *and neither of its edges horizontal; draw its plan and elevation.*

The projections of the square surface inclined 50°,

and no edge horizontal, must be first determined by the construction explained in preceding problems. The cube can then be built upon it.

A B C D is the figure first drawn; all the sides making angles with the h. t. Its plan is *a b c d*, and elevation *a′ b′ d′ c′*.

At the four corners of the square, perpendiculars to its surface must be raised.

It was shown in Problem XXXI., that if a line be perpendicular to a plane, its projections are perpendicular to the traces of that plane. The elevation of these perpendiculars, therefore, will be lines drawn through *a′ b′ c′ d′* at right angles to *v′ f′*. The lengths of these elevations will be equal to the edges of the cube, as those edges are parallel to the v. p. Make *a′ e′* and *c′ g′* equal to A B and join *e′ g′*. Then the elevation will be complete (*b′ f′* being a dotted line). To complete the plan, draw *b f*, *c g*, *a d* and *e h* parallel to X Y, and determine the points, *e f g h* by projectors through *e′ f′ g′ h′*. A glance at the position of the solid as represented by the elevation will show the student that the point A will be hidden in the plan. The edges *a e*, *a b*, and *a d*, which radiate from this point, must therefore be dotted. A horizontal section of the cube is shown in the elevation by the line *k′ t′*. The plans of the several points of this section are determined by projectors through the elevations of those points, intersecting the plans of the edges upon which they occur. Notice that the point *t′* is the elevation of two points of section—one, upon the edge C D, and the other upon B C. These will be shown separately in plan in *p* and *t*.

As the section is a horizontal one, its plan is also its true shape.

Problem XXXIX.

An Equilateral Triangle, A B C, *is inclined* 40°; *the edge,* A B, *of the figure is horizontal and* ·5″ *above the paper; draw plan and elevation.*

Commence this problem by finding a line upon a plane inclined 40°, ·5″ above the paper, and horizontal. The line *k a′*, parallel to X Y at a distance ·5″ from it and intersecting the v. t., gives *a′*, the elevation of such a line.

Construct this point into the h. p. by the arc *a′* A′, and draw A B parallel to *f h*. Upon it construct the equilateral triangle, and proceed to find its plan when rotated into the plane *v′ f h*, as before.

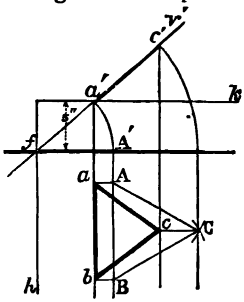


Fig. 128.

Problem XL.

A Prism 2″ long, its base being an equilateral triangle of ·5″ side, has one of its triangular faces inclined 30°, the edges of that face being each inclined to h. p.; draw plan and elevation.

The rectangle, A B C D, must be drawn with neither of its sides parallel to the h. t.

The projections *a b c d, a′ b′ c′ d′*, are determined as before. To complete the plan and elevation of the solid, we have to project the points G and K, which are the apices of the triangles forming its bases. G is perpendicularly over the centre of C B, and K over A D. Bisect, therefore, *b′ c′* and *a′ d′* in *e′* and *f′*. Draw *f′ g′* and *e′ k′* perpendicular to the v. t. The height of these points above

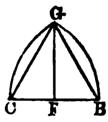


Fig. 129.

the base is the altitude of the equilateral triangles. This must be deduced from a supplementary drawing as shown in fig. 129. The length, F G, measured along $f' g'$ and $e' k'$, gives g' and k', the elevations of the points G and K. Join these to the proper points in the base,

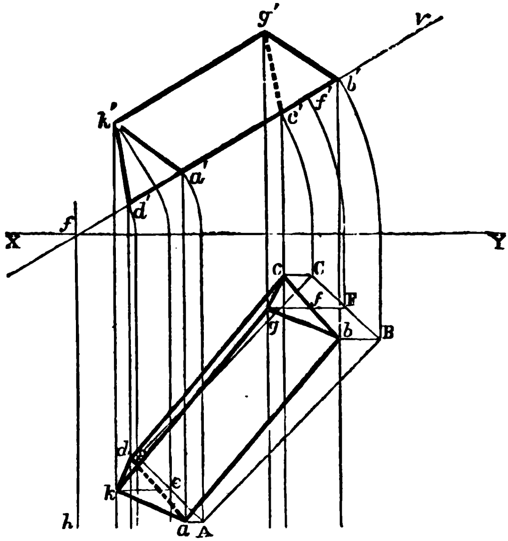


Fig. 130.

and the elevation of the solid will be complete. The plans g and k will be found upon lines perpendicular to the h. t. drawn through f and e, at the points where projectors through g' and k' meet them.

PROBLEM XLI.

A Tetrahedron has one of its faces inclined 24° to the h. p. and a line bisecting that face is horizontal; draw plan and elevation.

Draw an equilateral triangle, A B C, and show upon it a line B D bisecting it. Assume the h. t. of the oblique plane to be parallel to this line, and X Y perpendicular to it. The v. t. will therefore be drawn through *f* at an angle of 24° with the ground line. Rotate the figure into this plane and obtain its plan. The solid is then completed by projecting its axis. This is effected by finding the centre of the triangle, A B C, and the height of the tetrahedron, the construction for which is described in Chap. II. The point, D, can then be rotated into the plane *v′ f h*, and the elevation of the axis *b′ e′* determined.

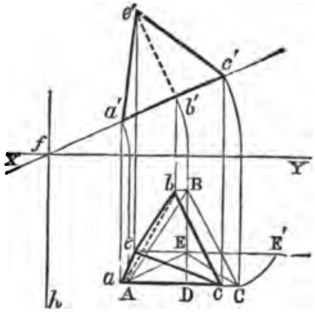


Fig. 131.

The plan of the apex falls in a projector through *e′* at the point where it is intersected by a parallel to X Y through E.

If two lines, making a certain angle with each other, are contained by an oblique plane, the plans of those lines will make a larger angle. This is easily illustrated by rotating a 45° set square upon its longest edge. The right angle will be projected in plan as a constantly increasing angle during the rotation, until the set square becomes vertical, when the plan is a straight line. The relation between the magnitude of the real

angle and that of its projections is sufficient to determine the inclination of the plane which contains the lines, and also that of the lines themselves.

If a figure rotate upon one of its edges, we have already found that the plan of any point in that figure will move in a line perpendicular to the axis of rotation.

We proceed, now, to discuss a few problems dependent for their construction upon the above principle.

Problem XLII.

The plan of a 3″ line is 2″ long. What is its inclination.

Draw a line 2″ long, and at one extremity, *b*, erect a a perpendicular. With the other extremity, A, as as centre, radius 3″, describe an arc, intersecting the perpendicular in *b′*. Then, A B will represent the line, and A *b* its plan. The angle between these two measures the required inclination.

Problem XLIII.

Two lines, each of equal length, meet at an angle of 60°, and are equally inclined. The plans of these lines contain a right angle. Determine the inclination of the plane containing them, and that of the lines themselves.

Draw two lines, A B and B C, meeting at an angle of 60°. As they are to be so placed as to be equally inclined, the axis of rotation must be equidistant upon A B and B C from B. Join A C, and assume that the triangle revolves upon it. Produce it in either direction and consider it as the h. t. of the required oblique plane. The ground line must then be taken perpendicular, to *h f*. The point B,

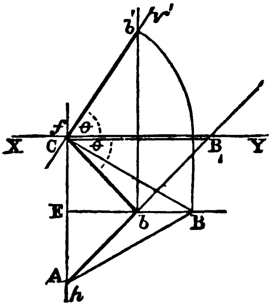


Fig. 132.

when the triangle revolves upon A C, will travel in an arc, whose plan is a perpendicular to *h f*. Draw B E, and determine point *b* in it, such that the angle A B C shall be a right angle. This is effected by describing a semicircle upon A C. Join A *b*, *b* C. Then the plan of the two lines, when B is raised into the required position, is shown.

The elevation, *b'*, of B will be in a projector through *b*, where it is intersected by the arc B' *b'*.

Furthermore, *b'* is a point in the v. t. of the required plane containing the two lines.

Join *b' f*, and the angle which it makes with X Y is the inclination of that plane.

To determine the inclination of either of the lines, set out *b* B perpendicular to one of the plans, and mark off from it a distance equal to the height of the point B, above the paper, as shown in the elevation. Then the angle *b c* B is the inclination of the lines.

By this construction the line B C is brought into the h. p.

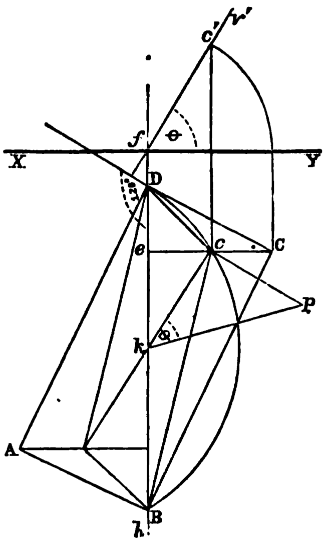


Fig. 133.

PROBLEM XLIV.

A rectangle A B C D, *revolves upon one of its diagonals until the plan of one of the opposite right angles contains an angle of* 120°. *Determine the inclination of the figure and that of the other diagonal.*

The construction of this problem is based upon the same principles as the preceding, but is rather more complicated.

The rectangle, A B C D, being drawn, the diagonal B D is assumed as the h. t., or axis of rotation. The plan of C will be in C *e*, drawn perpendicular to this diagonal, in such a position that the angle D C B, shall contain 120°. To discover this point, a segment of a circle containing that angle, and having B D for its chord, must be determined. (The method of construction is shown in fig. 36, *Plane Geometry*).

The elevation of C gives one point in the v. t., and the angle $v' f y$ is the inclination of the plane of the rectangle.

If C be joined to *k*, *c k* will be the plan of half the remaining diagonal.

The inclination of this line is shown by the angle C *k* P—and this is necessarily that of the whole diagonal.

EXERCISES.

1. A square has its surface inclined 40°, neither of its sides being horizontal. Draw plan and elevation.

2. A rectangle, 2" long by 1" broad, is inclined 50° to the paper, one of its diagonals being horizontal. Draw plan and elevation.

3. A circle of 3" diameter lies in a plane inclined at 55° to the paper. Draw a plan and an elevation of it, the ground line making an angle of 60° with the horizontal of the plane.

Note.—The second elevation is drawn upon a vertical plane having its X Y making an angle of 60° with the horizontal trace of the oblique plane.

4. A square prism, base 1·5" by 4" long, has one of its rectangular faces inclined 40°, the diagonal of that face being horizontal. Draw plan and elevation.

5. An isosceles triangle, whose vertical angle is 30° and base 1·5", revolves upon that base so that the plans of its sides are at right angles. Determine the inclination of the plane of the triangle and of the sides.

6. A regular pentagon of 1·5" side lies in a plane inclined at 70°, one *side* being horizontal. Draw its plan and elevation, the ground line being parallel to one of the sloping diagonals.

7. A pyramid having a square of 3" side for its base, and its height half the diagonal of the square, is to be shown by its

plan and elevation when one side of the base is horizontal, and the plane of that base is inclined 50° to the paper.

8. An equilateral triangle of 3″ edge revolves upon one of its sides until the plan of the opposite angle is 80°. What is the inclination of the figure and of the other two sides?

9. A cylinder, base 1″ radius, 3″ long, has its base inclined 50°. Draw plan and elevation, and determine the true shape of a vertical section of the solid passing through the middle of its axis.

10. A hexagon is inclined 40°, neither of its diameters is horizontal. Draw plan and elevation.

11. A pentagon revolves upon one of its diagonals until the angle opposite is shown in plan as 120°. What is the inclination of the figure and that of the other diagonals?

12. The axis of a square pyramid, base 1″ side, 4″ long, is inclined 60°, one edge of the base being horizontal. Show true shape of a horizontal section bisecting the axis.

13. An octahedron, 2″ edge, has one edge horizontal and an axis inclined 30°. Draw plan and elevation.

CHAPTER VI.

ON THE INTERSECTION OF PLANES AND LINES.

If two planes meet one another, their intersection is a straight line, but if a line intersects a plane it does so in a point.

The angle between two planes which intersect is their *dihedral* angle, and is measured by the angle between the two lines, which would be made by a third plane cutting them perpendicular to their intersection.

PROBLEM XLV.

To determine the Plans and Elevations of the intersections of the given planes. Cases 1, 2, and 3.

In the first case, the planes slope in opposite directions.

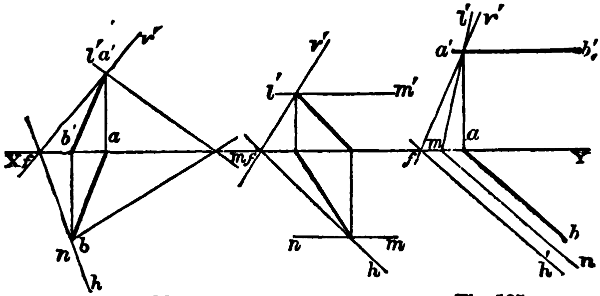


Fig. 134. Fig. 135.

The points a' and b, where the horizontal and vertical traces meet are in the intersection.

A is in the v. p. and B in the h. p.

The plan of A and the elevation of B are therefore upon X Y. Through *a'* and *b* draw projectors *a' a* and *b b'*. Join *a' b'* and *a b*. These, then, are the projections of the intersection of the given planes.

The inclination of the intersection can be determined by "constructing" it into the v. p. To do this, take *a* as centre, radius *a b*, and describe the arc *b* B, meeting X Y in B. Join *a'* B and the angle which it makes with X Y is the inclination of the intersection.

In the second case, the plane, *v' f h*, having its traces meeting in X Y, is intersected by another plane *l' m*, *m n* having its traces parallel to X Y.

The construction is similar to that of the first case, and needs no special explanation.

In the third case, the planes which intersect have their horizontal traces parallel to each other. In such a case the intersection is a horizontal line passing through the point where the vertical traces meet, and parallel to the horizontal traces.

Through *a'* draw a projector to determine *a*, its plan. Then, the elevation of the intersection is *a' b'* drawn through *a'* parallel to X Y, and the plan is *a b* drawn through *b*, parallel to the horizontal traces.

In fig. 136 the method is shown for determining the intersection of the planes *v' t'* and *l' m'*, *m n*, both of them being parallel to X Y. This forms a fourth case, requiring a special construction.

The student will see that, if the planes be themselves parallel, there can be no intersection, but if not, the intersection is a line parallel to both co-ordinate planes.

Conceive a right-angled triangle perpendicular to the v. p. to fit under each of the inclined planes; the bases of these triangles to be in the h. p., the perpendiculars in the v. p., and the hypotenuses in the oblique planes. These latter will cross each other in a point which is in the intersection of the planes.

Again, conceive these triangles to be constructed into

each of the co-ordinate planes in turn, the axes of rotation being respectively their perpendiculars and bases.

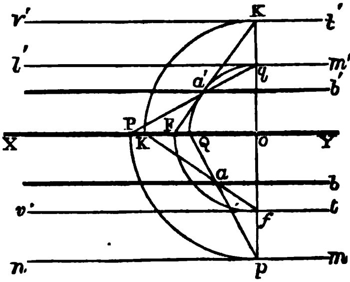


Fig. 136.

Then the points where one hypotenuse intersects the other, as shown upon these planes, will be the plan and elevation of one point in the intersection required. Draw K p perpendicular to X Y, meeting it in O. With O as centre, and radii O p and O f, describe the arcs p P and f F, intersecting X Y in P and F, Join P q and F K. Then the line a' b' through a', and parallel to X Y, is the elevation of the intersection.

The converse construction as shown in the diagram determines b, a point in the plan, and a b, drawn through a, parallel to X Y, completes it.

Problem XLVI.

To determine the projections of the intersection of a Line A B, and a Plane, v' f h.

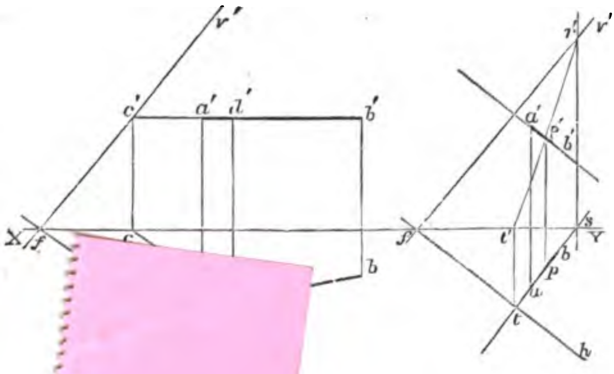
In the first instance, we will suppose the line A B to be horizontal, as in fig 136. It is clear that A B, being parallel to the h. p., will intersect the given plane in a point which is the same height above X Y as the line itself.

Produce the elevation a' b' beyond a' until it meets v' f in c'. Determine c, the plan of this point, and through it draw c e parallel to f h.

This line contains the plans of all points contained by the oblique plane, which are at the same height above the h. p. as the line A B.

The point d, where c e meets a b, is the plan of the intersection of the line and plane. Its elevation is d'.

When, as in the second case, the line is inclined to both the co-ordinate planes, a different construction is necessary. A vertical plane must be assumed con-



. 137.

tain the intersection of this plane
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ation $r'\,t'$ of the intersection of
e construction described in a pre-
point p' where $r'\,t'$ meets $a'\,b'$ is
intersection of the line A B with
its plan.

OBLEM XLVII.

To determine a Line perpendicular to the plane $v'\,f\,h$,

passing through a given point p′ p, and to show its intersection with the plane.

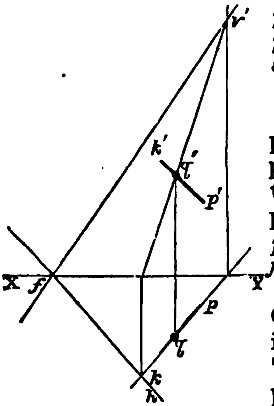

Fig. 138.

Lines perpendicular to a plane have their projections perpendicular to the traces of that plane. Through the points p' and p, draw $p'\,k'$ and $p\,k$ perpendicular to $v'f$ and $f\,h$.

Determine the intersection Q, of the line and plane as in the preceding problem. Then the line, P Q will be perpendicular to the plane, and have one of its extremities contained by it.

Problem XLVIII.

The point p, fig. 139, is the plan of a Point contained by the oblique plane v′ f h. It is required at the point, P, to raise a perpendicular to the given plane 1·5″ long.

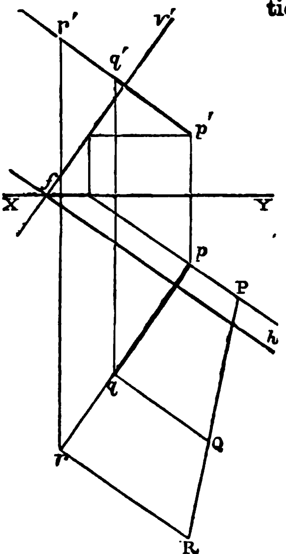

Fig. 139.

Determine first the elevation of the point P, by the construction described in Prob. XXIX. Through p' and p draw $p'\,r'$ and $p\,r$ at right-angles to $v'\,f$ and $f\,h$. This line P R must then be constructed into the h. p.

Assume therefore any second point, R, in it, and construct the points P and R into the h. p., as shown in the diagram. From P in P R mark off P Q – 1·5″ long

(the length required). Then if the point Q be projected back on to the h. p. as q, and the elevation q' be determined; the lines $p' q'$ and $p q$ will be the projections of a line P Q 1·5″ long perpendicular to the plane $v f h$, and passing through the point P.

Problem XLIX.

To determine the Angle between the two given planes, $v' f h$ and $l' m n$.

The plan and elevation $a b$, and $a' b'$ of their intersection must be first determined. A plane, perpendicular to A B, must then be assumed, and the lines which this plane will make by its intersection with those given, will meet at the angle required.

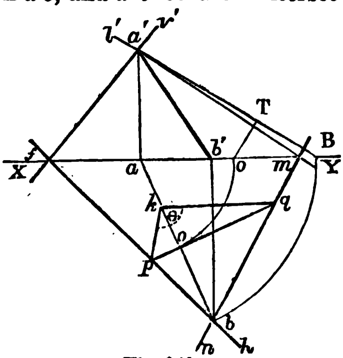

Fig. 140.

Further, the intersection of the assumed plane with the two given ones and the h. p. will form a triangle, and if this figure be constructed into the h. p., its vertical angle will measure the dihedral angle.

Draw $p q$, perpendicular to $a b$, and meeting the horizontal traces of the given planes in p and q, and intersecting $a b$ in o. This line, $p q$, is the base of the triangle mentioned above. With a as centre, radii $a o$, and $a b$, describe the arcs b B and o O, and join a' B.

By this means we construct the intersection of the two planes into the v. p. The altitude of the triangle is measured by a line, O T, passing through O, perpendicular to a' B. From the point O, in $a b$, mark off a

distance, $o\ k$, equal to O T, and join $k\ p$, $k\ q$. Then the angle, $p\ k\ q$, will be that between the given planes.

There are certain conditions, according to the relative position of the planes, under which this problem would require a modification of the construction for its solution.

For instance, the line $p\ q$, perpendicular to the intersection, may be parallel to one of the horizontal traces. In such a case, one point, as q, of the triangle would be infinitely distant. Then the line, $k\ q$, must be drawn parallel to $a\ b$, and the angle, $p\ k\ q$, would determine the dihedral angle as before.

Problem L.

To determine a Plane perpendicular to the given line, A B, *and passing through the point,* C, *in that line.*

The traces of the required plane will be perpendicular to the projections of the given line, and as the plane is to contain the point, C, it will also contain the horizontal, passing through C, which is perpendicular to A B.

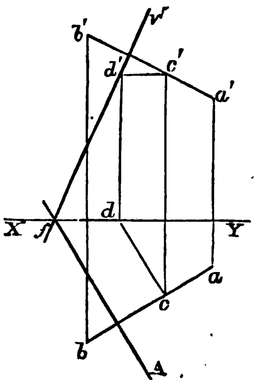

Fig. 141.

Through c draw $c\ d$, perpendicular to $a\ b$, meeting X Y in d. Determine d', the elevation of d, by a projector through d meeting a parallel to X Y, through c'. Then $c'\ d'$, $c\ d$ are the projections of a horizontal line contained by the required plane, and d' is a point in the v. t. Draw therefore $v'\ f$ through d' perpendicular to $a'\ b'$, and $s\ h$ through f perpendicular to $a\ b$.

PROBLEM LI.

To determine the angle contained by two Straight Lines, A B *and* B C, *given by their projections.*

If the horizontal traces of the lines be joined, a third line will be formed, which, with those two given, will complete a triangle, the vertical angle of which is that required in the problem. If the triangle be constructed into the h. p., its base being the axis of rotation, its true shape will be determined, and consequently the required angle between the lines, A B, B C.

Determine *d* and *e*, the horizontal traces of A B and B C. Join *d e*. Then *d* B *e* is the triangle mentioned above. In constructing it into the h. p., *d* and *e* will be stationary, and the point, B, will travel in a vertical plane, perpendicular to *d e*.

Through *b* draw *b f*, perpendicular to *d e*, and produce it beyond *b*. The actual distance of point B from *f* is the length of the hypothenuse of a right-angled triangle, of which *b f* is the base, and *o b′* the perpendicular. Set off from *o*, along X Y *o f′*, equal to *b f*. Join *b′ f′*, and make *f* B in the plan equal to *b′ f′* Join B *d* and B *e*, and the angle, *d* B *e*, is that between the two given lines.

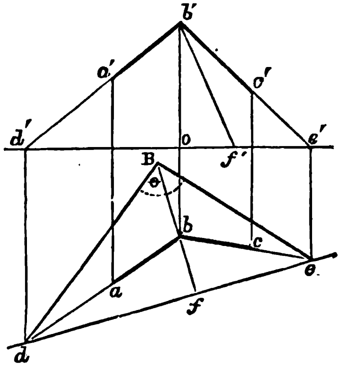

Fig. 142

CHAPTER VII.

ON THE PROJECTIONS OF FIGURES, HAVING GIVEN THE INCLINATIONS OF THEIR SURFACES, AND THAT OF A LINE BELONGING TO THEM.

It will be advisable at this point to recapitulate a few of the facts, mentioned in chapters IV. and V., upon lines lying in oblique planes.

It was there shown that if a line be contained by an oblique plane, it could only be horizontal when it was parallel to the h. t. of that plane. If such a line be perpendicular to the h. t., it will be inclined fully as much as the plane itself, and should it occupy any position upon the plane between these two, it will be inclined less than the plane. Thus we see, that the inclination of a line cannot exceed that of the plane which contains it. In this chapter, it will be our duty to treat of the projection of figures which lie upon oblique planes, their exact position upon them being fixed by the inclination of some line belonging to them.

Thus, if a square has its surface inclined at 40°, and one of its sides 20°, its position with regard to the co-ordinate planes is defined.

Problem LII.

To determine the projections of a line inclined 20°, *contained by a plane inclined* 40°.

It is most convenient, in drawing the traces of the

oblique plane, to assume the h. t. perpendicular to X Y. Let *v′ f h* be the traces of the plane inclined 40°.

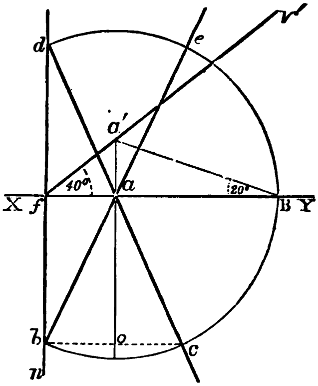

Fig. 143.

Assume a point *a′* in the v. t., and set out *a′* B, making an angle of 20° with X Y. Draw *a′ a* perpendicular to X Y. With *a* as centre, radius *a* B, describe the arc B *b*, intersecting the h. t. in the point *b*. Join *a b*. Then *a b* is the plan of the required line.

The first line *a′* B is in the v. p., and when the arc B *b* was described, one extremity, B, was brought round until it rested upon the h. t. of the oblique plane, the other extremity, *a′*, being fixed. We are sure, therefore, that *a b* is the plan of a line contained by the plane *v f′ h*, and it is inclined 20°, as its plan-length bears the same relation to the line itself as *a* B to *a′* B.

The elevation *a′ b′* is shown upon the v. t.

A second line, A C, is indicated in the figure, which has the plan, *c*, of one of its extremities on the other side of the horizontal line, *a′ o*, and at the same distance from it as *b* in *a b*. This line has the same inclination as the former one, and passes through the same point A.

We learn, therefore, that two lines can be contained by an oblique plane, equally inclined to the h. p. and passing through the same point.

If these two lines be produced towards *d* and *e*, the portions A D and A E will be in the second dihedral angle, the point *d* being equally distant with *b* from *f*. The lines A D and A B slope in opposite directions; in fact, the point *d* can be determined by

continuing the arc B b until it meets the back portion of the h. t.

PROBLEM LIII.

Given the plans of Lines, lying in an oblique plane, to determine their true lengths and inclinations.

Let $a\,b$, $a\,c$, be the plans of two lines, A B, A C, contained by the oblique plane $v'\,f\,h$. The elevation will in both cases be $a'\,f$. In Prob. VIII. a method was given of determining the true length and inclination of a line when its projections are known, and the construction there adopted will solve the above problem, but a second method, which is preferable in this case, is shown in the diagram.

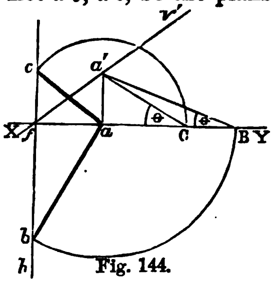

Fig. 144.

With a as centre, radii $a\,c$ and $a\,b$, describe the arcs c C and b B, intersecting X Y in B and C.

Determine a', the elevation of A, and join a' C and a' B. Then the angles which these lines make with X Y are the respective inclinations of A B and A C. The lines a' B and a' C also represent their true lengths.

PROBLEM LIV.

Draw the plan and elevation of Two Lines (any length) lying in a plane inclined 50°, and meeting in a point. The one is inclined at 25°, the other at 35°; determine the

true angle between them. The lines are to slope in opposite directions.

Commence by drawing *v′ f h*, the traces of a plane inclined 50°. Assume *a′* as the point where the two lines meet, and set out *a′* B and *a′* C, making respectively angles of 25° and 35° with X Y.

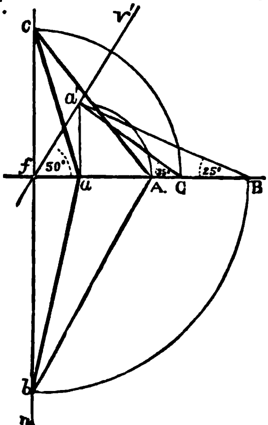

Fig. 145.

Determine *a*, the plan of *a′*, and as the two lines are to slope in opposite directions, with *a* as centre, radius *a* B, describe the arc B *b*, meeting the h. t. in *b*, and with the same centre, radius *a* C, describe the arc *c* C meeting the h. t. in *c*.

Join *a b* *a c*, and these lines will be the plans of those described in the question.

To determine the angle between them, construct *c* A *b* into the h. p. upon the h. t. The point A will travel in the v. p., and will, therefore after rotation, be in X Y. With *f* as centre, radius *f a′*, describe the arc *a′* A, intersecting X Y in A.

Join A *b*, A *c*, and *b* A *c* will be the required angle between the lines.

Problem LV.

A Square, C D E F, *has its surface inclined* 50°, *the side*, C D, *being inclined* 24°; *draw plan and elevation.*

In the first place, a line inclined 24°, and contained

by a plane inclined 50° must be determined. This has been described in the preceding problems. *a b* is the plan of such a line.

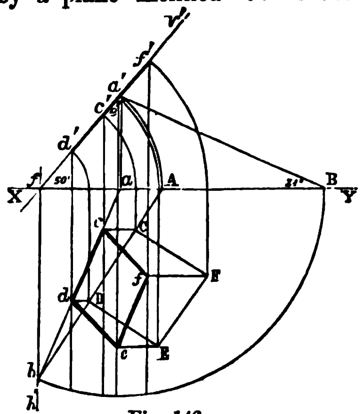

Fig. 146.

When this has been done, the line must be constructed into the h. p., and the square built upon it so that one side shall coincide with the constructed line. The square will then be in such a position that when it is revolved back into the oblique plane, the side which coincides with A B will be inclined 24°. With *f* as centre, radius *f a′*, describe the arc *a′* A. The point *b* will remain stationary. Join A *b*. If the problem be worked correctly, it will be found that the lines *a′* B and A *b* will be equal in length, and, upon consideration, we shall see that it should be so, as the former is the line, A B, constructed into the v. p., and A *b*, that same line, into the h. p.

Upon A *b* (at any part of it) construct the square, C D E F; rotate it back into the oblique plane, and determine its plan. This latter operation is fully described in Chap. V., and will present no difficulty.

The plan, *c d*, of the side, C D, should fall upon *a b*, proving that it is inclined as required by the problem.

Problem LVI.

To draw the plan and elevation of a Hexagon whose surface is inclined 50°, *one of its diameters being inclined* 20°.

This problem is introduced to show that the line, the inclination of which is given, need not be a side of a

figure. It may be any line connected with it. Proceed as before to determine the plan and elevation of a line inclined 20°, contained by a plane inclined 50°. *a b* is the plan of such a line. Construct it into the h p., and *a* B will represent its true length.

Arrange a hexagon about this line so that its diameter may be upon *a* B. To do this, take any point, *o*, as

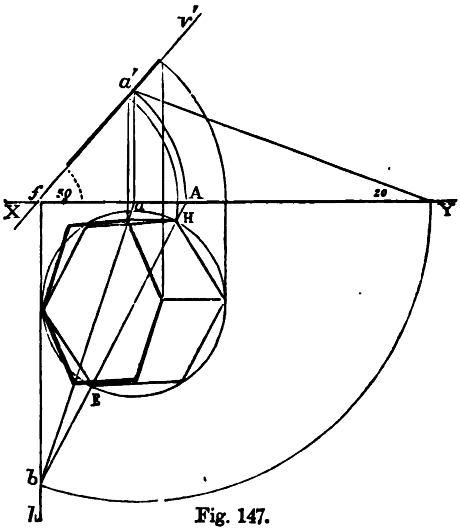

Fig. 147.

centre, and with a radius equal to the side of the required hexagon describe a circle. Then E and H, where this circle intersects *a* B, will be two points in the figure. Finish the hexagon and rotate it into the oblique plane, determining its plan as before.

Problem LVII.

The plane of a Pentagonal Figure of 1″ *side is inclined* 50°.

The plan of one of its sides is ·75″ long; draw its plan and elevation.

This problem presents a slight modification of those preceding it. Instead of telling the student the inclination of a line in the figure, the relation between its true length and its projection upon the h. p. is given. The first thing, therefore, to be done is to determine at what angle a line 1″ long must be inclined, so that its plan may be ·75″ long. This is shown in a supplementary drawing in fig. 148.

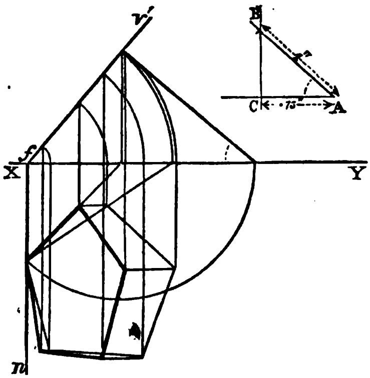

Fig. 148.

Take any line, A C, ·75″ long, and at one extremity, as C, erect an indefinite perpendicular, C B. With the other extremity as centre, and 1″ as radius, describe an arc meeting A B in B. Then the angle which A B makes with A C is the inclination of the line.

The problem now resolves itself into that of projecting a pentagon whose surface is inclined 50°, and one of its sides at an angle equal to θ. The construction of this needs no further explanation.

Problem LVIII.

A Cube of 1·5″ *edge is to be shown in plan and elevation when the plane of one face is inclined* 50°, *and one edge of that face is inclined* 30°.

Commence by determing the plan of the face, the position of which is given. This will present no

difficulty. Upon the elevation show the edges of the solid perpendicular to that face, and also the straight line which represents the opposite face. The plan is

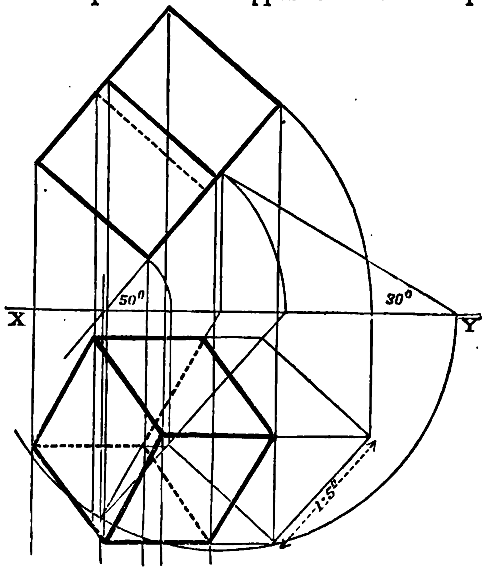

Fig. 149.

finished by determining those of the four points of the upper face and joining them to the proper points, which are already obtained.

Problem LIX.

Given a b, the plan of a Line, A B, *contained by a plane neither of whose traces is perpendicular to* X Y; *required the elevation.*

When a line is contained by a plane, the traces of the line are in the traces of the plane. If, then, the

plan a b be produced beyond a until it meets $f h$ in d, that point will be the h. t. of the line A B, and if it be produced beyond B until it meets X Y in e, and a perpendicular through e be drawn, intersecting the v. t. in e', that point will be the v. t. of the line A B.

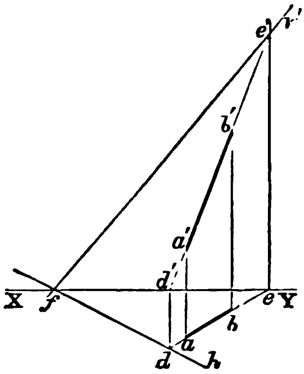

Fig. 150.

The point d' is the elevation of d, and d' joined to e' will give $e' d'$, which is the elevation of the line A B produced in each direction.

Projectors through a and b will determine $a' b'$, the part of $d' e'$ which is the elevation of A B.

Problem LX.

To determine the projections of a Line inclined 20° contained by the plane $v' f h$.

The line cannot be inclined more than the plane which contains it.

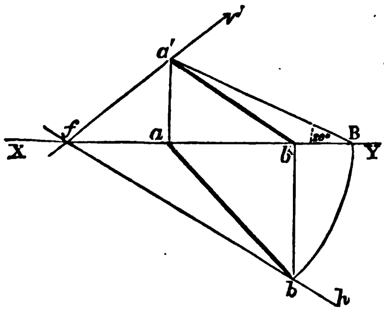

Fig. 151.

Assume a point a' in the v. t. of the given plane, and draw a' B, meeting X Y at an angle of 20°. This line is the elevation of one inclined 20°, contained by the v. p. Determine a, the plan of a', and with a as centre, radius a B, describe the arc B b, intersecting the h. t. in

the point b. Find b', the elevation of b, and join a to b and a' to b'. Thus the line A B will be shown by its projections a' b' and a b.

EXERCISES.

1. Draw the plan and elevation of a line inclined 40°, contained by a plane inclined 80°.

2. Draw the plan of a square of 3″ side when its plane is inclined at 63° and one side at 25°; add an elevation, the ground line being parallel to the shortest diagonal in plan.

3. A hexagon, 1″ side, has its surface inclined 40°, one side being inclined 20°. Draw plan and elevation.

4. A pentagon has its surface inclined 50°, one diagonal being inclined 30°. Show plan and elevation, and state the inclination of each of the other sides.

5. A right angled triangle, 3″, 4″, and 5″, lies in a plane inclined 40°. Its hypotenuse is inclined 20°. Draw plan and elevation, and state the inclination of the base and perpendicular.

6. A square pyramid, base 1″ edge, 3″ long, is to be drawn when its base is inclined at 47°, and one edge of that base at 27°.

7. A square prism, base 1″ side, 3″ long, has its axis inclined 60°, one edge of the base is inclined at 10°. Draw plan and elevation, and show the true shape of a section made by a vertical plane bisecting the axis at an angle of 40°.

8. A triangle, A B C, A B, 3″, B C, 3·5″, and A C, 4″, lies in a plane inclined 70°, the line joining the middle point of A B to C is inclined 20°. Draw plan and elevation.

9. A tetrahedron, 2″ edge, has one face inclined 50°, whilst the line bisecting this face is inclined 40°. Draw plan and elevation.

10. An octahedron, 1·5″ edge, is so situate that the plane containing two of its diagonals is inclined 80°, one of these being inclined 45°. Draw plan and elevation of the solid.

11. A square, A B C D, 2″ side, is inclined 40°, the line joining A to the middle point of C D is inclined 15°. Draw plan and elevation, and determine the inclinations of A B and B C.

12. A cube, 3″ edge, has one of its faces inclined 60°, the plan of one of the edges of that face is 2·5″ long. Draw plan and elevation.

13. An equilateral triangle of 1″ side is the base of a prism 4″ long. One of its faces is inclined 45°, and an edge of the base belonging to that face is inclined 21°. Draw plan and elevation of the whole, and add a sectional elevation upon a vertical plane, bisecting the axis at an angle of 70°.

CHAPTER VIII.

ON THE PROJECTION OF OBLIQUE SURFACES, WHEN THE INCLINATION AND THE ANGLE BETWEEN TWO LINES IN THEM ARE GIVEN.

WE have already seen that, if the inclinations of a figure and a line belonging to that figure are given, the position is fixed as regards the co-ordinate planes. This is also the case if the inclination of two lines upon the figure, meeting at a given angle, are known.

Thus, if an equilateral triangle has two of its sides inclined, at angles of 20° and 30° respectively, its projections upon the co-ordinate planes will be the same wherever that triangle may be placed.

But the oblique plane which contains them cannot be assumed at once. It must be determined by construction.

It is well that the student should understand this at once, as in all problems upon inclined surfaces which have previously claimed his attention, he has been enabled to assume the oblique plane at the commencement. We proceed, therefore, to explain the determination of a plane containing two lines, having given their inclinations and the angle between them.

PROBLEM LXI.

Two Lines, A B, B C, *are mutually perpendicular; the former is inclined* 40°, *the latter* 20°; *required to determine by its traces the plane containing them.*

In the first place, the writer would recommend the student to cut out a piece of card-board of exactly the

same shape as that shown in fig. 152; that is, he must prick through upon some paper the points A, D′, B, D, C. Let him then cut through the lines *a* D′, D′ B, B D, D C, and C A. After he has done this, let him fold the model upon the lines B A and B C, until the points D′ and B coincide.

If he then lets the model rest upon the lines A D′, D C, and A C, he will have an inclined surface represented by the triangle, A B C; the edges, A B, B C, being also inclined to the h. p. The longer edge, A B, will be less inclined, whilst the shorter edge, B C, will be more inclined. The line A C will represent the h. t. of the oblique plane, which contains A B C, and consequently the lines A B, B C. Now, if the v. p. be assumed perpendicular to A C, the elevation of the triangle upon that plane will be a straight line. This straight line will be the v. t. upon the assumed v. p. of the inclined plane containing the triangle A B C. Now, referring to the problem before us, the following construction is necessary; the reason for drawing every line being perfectly clear, if it be investigated, by aid of the model described above:—

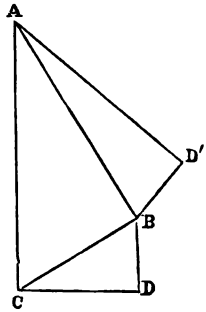

Fig. 152.

Draw any two lines, A B, B C (fig. 153), perpendicular to each other.* Assume a point, A in A B, and draw A D, making an angle of 20° (the angle at which A B is to be inclined) with A B. At the point B, make B D′ perpendicular to A D′. Then the distance, B D′, is the height to which the point B must be raised, in order that A B may meet the h. p. in A, at the given angle of 20°. Now, as the second line, B C, is to start from the same

* They are directed to be drawn perpendicular, because A B and B C are to be mutually at a right angle with each other. Should they have been set as meeting at any other angle, they should make that angle upon the drawing.

point, B, it is clear that the line B D must be made equal to A D′. But its direction is not yet known. To ensure the former condition, describe an arc, D′ D, with B as centre, and B D′ as radius. It follows then that, whatever be the position of B D, it must be a radius of the arc just drawn.

Furthermore, the line D C is to meet B C, at an angle of 40° (the angle at which B C is to be inclined).

Now, a line perpendicular to the radius of a circle is a tangent to it, therefore, D C must be drawn touching the arc D′ D and meeting B C in C at an angle of 40°.

The best method of doing this is to assume any line as E F (at 40° with B C) and, by means of the set square, to draw a parallel to E F tangent to the arc. It is unnecessary to draw the line B D.

We shall now have found two points, A and C, in the surface containing the two given lines which we can assume as being upon the h. p.

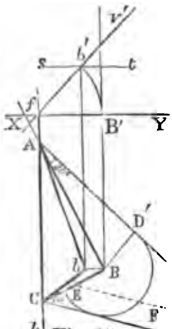

Fig. 153.

We will therefore join them, and consider the line A C as the h. t. of the oblique plane required. Our X Y must be taken perpendicular to A C. To determine the v. t., the triangle A B C must be rotated upon the side A C until the point B reaches a height equal to B D′. Then, the elevation of that point will be in the v. t. required. Through B draw a projector B B′, and, with *f* as centre, *f* B′ as radius, describe the arc B′ *b*′. Draw a line *s t* parallel to X Y, at a distance above it equal to B D′. This line will intersect the arc B′ *b*′ in *f*′, which is the elevation of B when in a position such that A B and B C make the given angles with the h. p.

If *b*′ be joined to *f*, the line *b*′ *f* will be the elevation of the whole triangle A B C, and its direction is that of the v. t. of the plane to be determined.

The plan of B will fall in a line perpendicular to the

h. t., its exact position being found by a projector through b'. Join b C, b A, and the lines thus formed will be the plans of A B, B C, when inclined as described in the problem.

Note.—The sum of the inclinations of the lines, together with the angle included between them, must not exceed 180°. When it equals that angle, the plane containing them is perpendicular to the h. p. The student will readily understand this when he remembers that the three angles of a triangle together make 180°, and that the lines will have their greatest inclination when the plane containing them is vertical. The angle between them is in that case the vertical angle of the triangle.

Problem LXII.

Two sides of an equilateral triangle are inclined at angles of 34° *and* 40° *respectively; determine the plan and elevation of the figure.*

The plane must be determined containing two lines meeting at an angle of 60° (the angle of an equilateral triangle), and inclined as directed in the question by the construction explained in the preceding problem. Commence by drawing the equilateral triangle and produce two of its sides in opposite directions *away* from the point where they meet, and treat them in the same way as A B and A C, shown in Prob. LXI. It is most convenient to assume the point

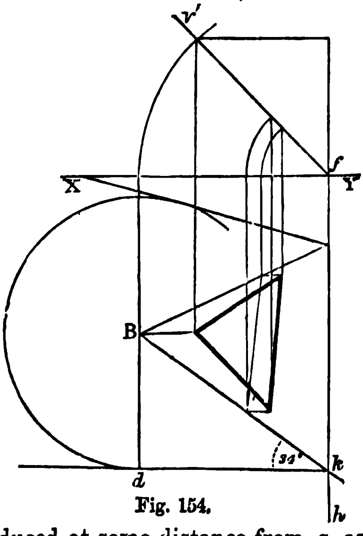

Fig. 154.

k in one of them, produced at some distance from a, as

by doing so the traces of the plane, when determined, will be clear of the figure. At the point k, make $k\ d$ at an angle of 34° with Bk, and proceed as before.

When the traces, $v'f$ and $f\ h$ of the oblique plane containing the triangle have been found, fold the figure into that plane, and thence determine the plan.

Problem LXIII.

Two Lines, A B *and* B C, *contain an angle of* 40°; *they are each* 2″ *in length; determine the plane containing them when they are so inclined, that the plan of* A B *is* 1·4″ *long, and that of* B C 1·6″.

This question is set in a similar manner to Prob. LVI. That is, instead of giving the inclination of two lines, the relation betwen their true length and that of their plans is stated. Take the length of the plan of A B, and at one extremity raise a perpendicular, with the other extremity as centre, radius 2″ (the length of A B), describe an arc intersecting the perpendicular. The angle contained by the base and hypotenuse of the triangle thus formed will be the inclination of the line A B.

Similarly, find the inclination of B C.

The problem then resolves itself into drawing the plan of two lines of given inclination, and containing a given angle between them.

Problem LXIV.

A Prism 2″ *long, whose base is an equilateral triangle of* 1″ *side, has two of the edges of that base inclined at angles of* 20° *and* 50° *respectively; draw its projections.*

Draw an equilateral triangle, and proceed, as in Prob. LXI., to determine the plane containing that triangle when two of its edges are inclined as described.

Determine the plan of the figure, and at the three points in elevation, which fall upon the v. t. of the plane, raise perpendiculars, each 2″ long.

Finish the elevation, and obtain the rest of the plan

by projectors, through the points of the base which is highest, meeting the plans of the perpendicular lines.

Notice that the plan of the upper triangle will be firm lines, and any dotted edges which occur in the plan can be reasoned out upon this assumption.

Problem LXV.

One diagonal of an Octahedron 1·5″ *edge, is inclined* 25°; *a second diagonal is inclined* 30°; *draw a plan and elevation of the solid.*

We have already learned that an octahedron has three diagonals or axes. Any two of these will also be the diagonals of one of the square sections of the solid.

Draw, therefore, a square of 1·5″ side, and show its diagonals. Consider that these latter are to be inclined as directed in the problem.

The plane containing the diagonals, and, consequently, the square, can be determined as in previous problems. The plan of the square, too, is readily deduced. There are two other points in the solid at the extremities of the third diagonal, which is perpendicular to the plane containing the square A B C D.

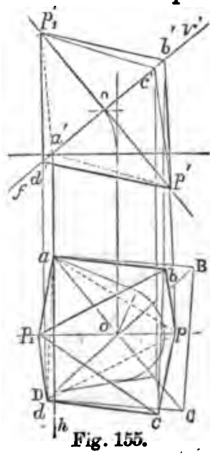

Fig. 155.

The projections of this line will be perpendicular to the traces of the plane $v' f h$. Draw, therefore, $p_1 p'$ perpendicular to $v' f$ and p, p—to $f h$ through the projections of the centre of the square.

The elevation $p_1 p'$ will be equal in length to the diagonal itself. Mark off therefore $o p$ and $o p'$ equal to O A. Join each of the four points a' b' c' d' to p_1 and p' dotting the hidden lines $a' p'$, and the elevation will be complete.

The plans of the points P and P′ will be determined upon a

line through O, parallel to X Y, by projectors through p' and p'.

The edges, which should be dotted in the plan, are best determined by noticing from the elevation which vertex of the solid is uppermost, and deciding accordingly.

Problem LXVI.

The three corners, A, B, *and* C, *of an Equilateral of* 1″ *side are* ·5″, ·75″, *and* ·3″ *above the paper respectively; determine the plane containing the figure and its plan.*

This is a modification of the cases explained in the previous part of this chapter. Instead of the inclination of two lines being given, we are told their lengths, and the heights of three points in them above the h. p. Such problems can be solved by one of two constructions—the more preferable of which, because the simpler, we proceed to explain.

Draw the equilateral triangle, A B C, and produce A B and B C, beyond A and C.*

With the points A B and C as centres, and with radii ·5″, ·75″, and ·3″, respectively, describe arcs as in the figure. Then a line, *p q*, tangent to the arcs which have A and B as their centres, will, when produced, meet the line B *n* in *n*. Similarly, a tangent, *r t*, to the arcs which have B and C for their centres, will meet B C produced in *k*. Then the angles, *p n* B, and *r k* B, are the angles of inclination of the lines A B and B C, when the points A B C are raised to the heights given in the problem.

The points *n* and *k* would be in the h. p., and the line, *n* K, joining them, is the h. t. of the plane containing the figure.

Take X Y, perpendicular to *k n*. To obtain the point b' in the v. t, draw a projector through B to meet X Y

* The line should be produced in directions *away* from the highest point.

in B′. Then with f as centre, radius fB′, describe the arc B′ b', and discover a point, b', in that arc which is ·75″ above X Y. Join $b' f$. Then $v' f h$ is the plane containing the figure when in the position required. Its plan and elevation can be determined in the usual manner.

The proof of the construction is made by noticing that the elevation of the three points, A B and C, are at their given heights above X Y.

When the *given heights* of any three points are considerable, so making the arcs, to be described in the construction, impracticable, the difficulty is remedied by subtracting an equal height from each of the given ones, and solving the problem with the new data thus obtained. Thus had A B C, in the present question, been 3·5″, 3·75″, and 3·3″ above the paper, the 3″ could have been subtracted from each, and the problem worked with the data ·5″, ·75″, and ·3″. The plans and elevations would have been the same as if the original numbers had been retained.

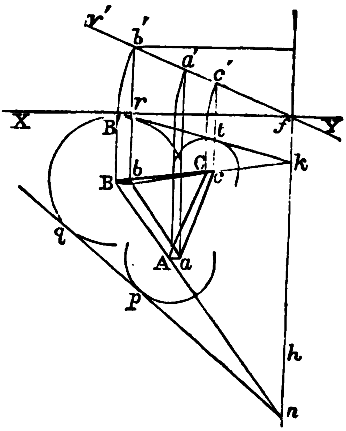

Fig. 156.

If the given height of the lowest point were subtracted from each, that point would be brought upon the h. p., as shown by the elevation.

Problem LXVII.

A Square, A B C D, 2″ *sides, has its corners,* A B *and* C, ·8, 1·2″, *and* ·3″ *above the paper respectively; draw its projections when in this position.*

This is a similar problem to the preceding, and can be

solved by the same construction. It is, however, our intention here to describe the second method of solution, previously mentioned.

The principle involved is that of finding some line upon the figure which shall be horizontal, when the corners are in the positions assigned to them in the problem. Upon this line, when determined, the figure is supposed to rotate until the points attain their specified heights. In fact, it is used as the h. t. of the plane of the inclined square. Commence by drawing the square, A B C D. The line B C will join the points of the figure which are to be the highest and lowest of those whose heights are given. It will be an inclined line, and there will exist upon it some point whose height above the paper will be ·8″ (that of the remaining given point). Seeing that the difference in height of the extremities of B C is ·9″ (1·2″ minus ·3″) and that A is ·5″ (·8″ minus ·3″) above C, we can infer that a point $\frac{5}{9}$ of the distance from C to B will be that unknown spot in B C whose height will be ·8″, and equal to that of A. Join A 5, and consider it as the horizontal line upon which the figure A B C D is to revolve, in fact, the h. t. of its oblique surface when in the required position. Assume X Y perpendicular to A 5, and through B draw the projector B B′, and with *a′* as centre, radius *a′* B′, describe the arc B′ *b′*. The point *b′* will be upon this arc at the height above X Y, which B is above A, that is, ·4″ (1 2″ minus ·8″). Join *b′ f* and consider it as the v. t. of

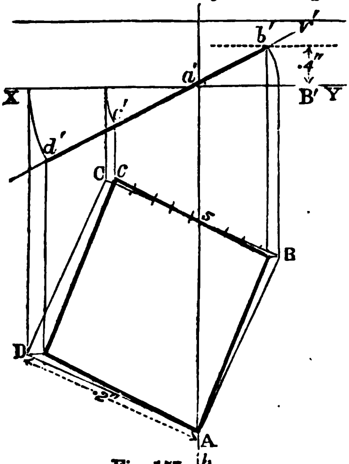

Fig. 157.

the plane, into which the square A B C D is to be revolved. The remainder of the construction will be understood by reference to the diagram.

Note.—By assuming A E as the h. t., the point C falls below X Y. This is immaterial, as the projections of the figure are the same as if the whole of it were above the h. p.

EXERCISES.

1. Two lines meet at an angle of 90°,—one is inclined 25°, the other at 40°. Draw plan and elevation.

2. A square, A B C D, of 3″ side is to be represented in plan and elevation, when the lines joining A B and the middle points of C D, A D are inclined at 30°.

3. A pentagon has two of its sides inclined 12 and 20° ; draw plan and elevation. State the inclinations of the other sides and the diagonals.

4. Draw the plan of a cube of 3″ edge, when two edges are inclined at 25° and 50° to the paper, and add an elevation on a ground line parallel to that diagonal of the cube which has the shortest plan.

5. Draw the plan and two elevations of an equilateral triangle A B C of 3″ side when the sides A B, B C are inclined at 35° and 55° to the paper (one elevation to be on a plane parallel to the line A C).

6. Two lines, each 3″ long, are at right angles ; draw the plan of them when one is inclined to the paper at 25°, the other at 50° ; add an elevation on a plane (*i.e.* the ground line) parallel to the line joining the extremities of the given lines.

7. A hexagon, 1″ edge, has its diagonal inclined 40°, and one side of the base, which meets this diagonal, 30°. Draw plan and elevation.

8. A tetrahedron, 2″ edge, has two of its sides making angles of 40° and 25° with the paper. Draw plan and elevation.

9. The plans of two adjacent sides of a square of 3″ edge are 2·5″ and 3″ in length ; draw a plan and elevation of the figure.

10. An equilateral triangle of 3″ side has one edge inclined 40°, the plan of an adjacent side being 2·5″ long. Draw plan and elevation.

11. The three corners of an equilateral triangle of 3·25″ side are raised above the paper 1″, 1·75″, and 3″ ; draw its plan in this position ; add an elevation on a ground line parallel to the shortest side of the plan.

12. A triangular pyramid, its four faces being equal equilateral triangles of 3″ edge, is to be shown in plan and elevation when three of its corner are 1″, 1·5″, 2·5″ above the paper.

SPECIMEN EXAMINATION PAPER, AS SET BY THE SCIENCE AND ART DEPARTMENT.

PLANE GEOMETRY.

1. Make a square equal in area to that of an equilateral triangle square, and hexagon, (each upon a base 1″ long), added together.

2. Draw a plain scale of yards and feet where a line 3·25″ long represents 27 yards.

SOLID GEOMETRY.

1. A line, A B, 2″ inch long, has one extremity, A, 1″ above the horizontal plane, and 1·5″ in front of the vertical plane. It is inclined 30° to the horizontal plane, and is parallel to the vertical plane. Draw plan and elevation, and determine its horizontal trace.

2. A hexagonal pyramid has its base inclined 40°, neither of the edges of that base being horizontal. Draw its plan and elevation.

3. A cube, 3″ edge, stands with a line joining two opposite corners, vertical. A plane perpendicular to the vertical plane cuts the solid. The section is shown in elevation by a line bisecting the vertical diagonal at an angle of 20°. Draw a sectional plan of the portion nearest to the ground line.

4. Show the projections of the intersection of the given planes, and in $v' f h$ place a line inclined 20°. State also the inclination of the intersection.

5. Draw plan and elevation of a tetrahedron, when two of its edges are inclined 40° and 20°.

6. A square, 3″ side, has its centre 2″ above the paper, whilst the heights of two of its corners are 1·8″ and 1·2″. Draw a plan of the figure.

7. An equilateral triangle, 2″ side, is the base of a pyramid 3″ high. Draw its plan and elevation, when the axis is inclined 30° and one edge of the base 10°.

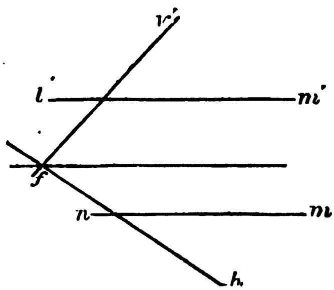

Fig. 158.

8. Draw a line, making an angle of 40°, with X Y, consider it as the horizontal trace of a plane inclined 60°. What is the angle between the traces?

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